Math 401: Sec 0401: Midterm # 2
Dec. 10, 2014

Problem 1: Consider $\mathbb{V} = \mathbb{P}_2$, i.e., the space of polynomials of degree ≤ 2 with the inner product

$$\langle p, q \rangle = p(-1)q(-1) + p(0)q(0) + p(1)q(1)$$

for all $p, q \in \mathbb{P}_2$.

(1) Prove that $\langle \cdot, \cdot \rangle$ defines an inner product on $\mathbb{V}$.

(2) Prove that $p(x) = 1$ and $q(x) = x$ are orthogonal with respect to $\langle \cdot, \cdot \rangle$.

(3) Find α and β such that
$$r(x) = x^2 + \alpha x + \beta,$$
is orthogonal to the set $\{p(x), q(x)\}$.

Problem 2: (1) Given the set of experimental data $(1,1)$, $(2,3)$, $(3,3)$, $(4,6)$, $(5,8)$, write the least squares problem $A\mathbf{x} \approx \mathbf{b}$ for fitting the data with a quadratic function

$$q(x) = a + b(x-1) + c(x-2)^2,$$

with unknowns coefficients a, b and c. Indicate how to find a, b and c, but **do not solve!**

(2) Using MATLAB, find the QR factorization of the associated matrix and solve the linear system.

Problem 3:

(1) Formulate the linear algebra problem of finding the closest polynomial $p \in \text{span}\{t, t^2\}$ to the function $f(t) = e^t \cos(t)$ with respect to the standard L^2 inner product. **Do not solve!**

(2) Is the matrix of (a) symmetric and positive definite?

(3) Using MATLAB, find the QR factorization of the associated matrix and solve the linear system.

Problem 4: (1) Find the Cholesky factor C of the following matrix:

$$A = \begin{pmatrix} 9 & -3 \\ -3 & 5 \end{pmatrix}$$

(2) Let $A = BB^T$. Under what conditions of B is the matrix A symmetric and positive definite?

Problem 5: (1) Find a set of orthogonal vectors that spand the same subspace $S \subset \mathbb{R}^4$ as

$$\mathbf{a} = (0, -1, 1, 0)^T, \qquad \mathbf{a} = (2, 0, 2, 0)^T, \qquad \mathbf{a}_3 = (-1, 0, 0, 1)^T.$$

(2) Find the projection of $\mathbf{b} = (-1, 1, 1, 1)^T$ onto S.

Problem 6: Let $\mathbb{V}$ be a vector space and $\mathbb{W}$ a subspace of $\mathbb{V}$. Consider the *orthogonal complement* $\mathbb{W}^\perp := \{\mathbf{v} \in \mathbb{V} : \langle \mathbf{v}, \mathbf{w} \rangle = 0 \quad \forall \mathbf{w} \in \mathbb{W}\}$.
(1) Prove that $\mathbb{W}^\perp$ is a subspace of $\mathbb{V}$
(2) Prove that $\mathbb{W} \cap \mathbb{W}^\perp = \{\mathbf{0}\}$.