Fall 2014 - Math 401 Section 0401
Applications of Linear Algebra

Matlab Project 2 - Due Date: December 10, 2014

You should complete each problem in a separate `m`-file and bring a copy of each problem to class. You can also use the new MATLAB© command `publish`.

Problem 1. (40 pts) *Data Fitting.* The following data represents the population (in millions) of the USA between 1900 and 1990:

t	1900	1910	1920	1930	1940	1950	1960	1970	1980	1990
y	75.995	91.972	105.711	123.203	131.669	150.697	179.323	203.212	226.505	249.633

Proceed as follows to find the least squares fit $p(t) = \alpha_0 + \alpha_1 t + \alpha_2 t^2 + \alpha_3 t^3$ to the given table using MATLAB©.

(a) Determine the rectangular matrix A and right-hand side $\mathbf{b}$ of the least squares problem.

(b) Form and solve the Normal Equations. Use the commands `A'` to transpose A and \ to solve the linear system. Plot the table and $p(x)$ using `plot(t,y,'.',x,p(x))`, where `x=[1900:0.01:1990]`.

(c) Let $B \in \mathbb{R}^{m\times n}$ with $n \leq m$, prove that $B^T B$ is symmetric and positive definite if $\text{rank}(B) = n$ (B is full rank). Next, use the command `rank`(A) to find the rank of A defined in (a). Compute the Cholesky decomposition of $A^T A$, using the command `R = chol(A'*A)` (see Olver-Shakiban p.168), and then solve the system by backward and forward substitution (use the command \).

(d) Repeat parts (a) and (b) with $q(t) = \alpha_0 e^{\alpha_1 t}$. Compare $p(t)$ and $q(t)$. Which one is the best approximation in the the Euclidean norm? Type `help norm` (Hint: see Olver-Shakiban Example 4.11, p.199).

Problem 2. (30 pts) *Data Fitting and Orthogonal Polynomials.* This problem repeats Pb#1 but replacing the canonical basis $\{1, t, t^2, t^3\}$ of $\mathbb{P}^3$ by orthogonal polynomials.

(a) Obtain orthogonal polynomials $\{p_i(t)\}_{i=0}^3$ with respect to the scalar product $\langle p, q\rangle = \int_a^b p(t)q(t)\,dt$ where $a = 1900$ and $b = 1990$. First, determine by hand the orthogonal basis of $\mathbb{P}^3$ on the interval $[-1, 1]$ by using of the Gram-Schmidt procedure. Then transform the derived basis to the interval [1900,1990] by the simple change of variables $x = (t - 1945)/45$. Explain why the resulting polynomials are still orthogonal.

(b) Repeat items (a) and (b) of Problem 1 using the orthogonal basis obtained above. Use the command `cond(A'*A)` to find the condition number of $A^T A$ and compare with that in Pb#1 (b). Draw conclusions.

Problem 3. (30 pts) The MATLAB© command `polyfit` returns the coefficients c_i for a polynomial of degree n

$$p(x) = c_1 x^n + \cdots c_n x + c_{n+1}$$

that is a best fit (in a least-squares sense) for the data $(x_1, y_1), \ldots, (x_m, y_m)$. In particular, when $n = m - 1$, $p(x)$ is the interpolating polynomial of degree $\leq n$ for the given $n+1$ nodes.

1. Consider the data $(x_i, \sin(x_i))_{i=1}^{10}$ with $x_i = 0, 1, \ldots, 10$.
   (a) In one graph, plot the data and the best polynomial approximation of degree 5 and 10 in the least-squares sense. To do this, use `polyfit` to find the best approximation and `polyval` to evaluate the polynomial. Type `help polyfit` and `help polyval` to learn how these commands work.
   (b) Verify that the polynomial of degree 10 interpolates the data (Hint: use the `norm` command).
2. Plot, in one graph:
   (a) the function $f(x) = \dfrac{1}{1+x^2}$ for $-5 \leq x \leq 5$;
   (b) the points $(x, f(x))$ for $x = -5, -4, \ldots, 4, 5$;
   (c) and the polynomials of degree 6 and 10 that fit the points in the least squares sense (use `help polyfit` and `help polyval`). Explain the cause of the observed oscillations.