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Numerical approximation of the LQR problem in a strongly damped wave equation

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Abstract The aim of this work is to obtain optimal-order error estimates for the LQR (Linear-quadratic regulator) problem in a strongly damped 1-D wave equation. We consider a finite element discretization of the system dynamics and a control law constant in the spatial dimension, which is studied in both point and distributed case. To solve the LQR problem, we seek a feedback control which depends on the solution of an algebraic Riccati equation. Optimal error estimates are proved in the framework of the approximation theory for control of infinite-dimensional systems. Finally, numerical results are presented to illustrate that the optimal rates of convergence are achieved.

Keywords Optimal control · Feedback control · Wave equation · Convergence rates · Finite element method

1 Introduction

In this paper we are concerned with the numerical approximation of an optimal control problem in a strongly damped equation. We consider a quadratic cost functional and a control law defined by means of a feedback operator acting over the system states. The so-called LQR (Linear-quadratic regulator) problem constitutes a cornerstone of the modern linear control theory. Studied originally in a finite-

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dimensional context, the LQR problem became also a subject of interest in the framework of control theory for partial differential equations (see [2, 12]), which are related to several applications, including the control of parabolic systems like the heat equation (see [5]), the active control of noise (see [6, 17]), and the active control of flexible structures (see [11, 13, 18]), among others.

In general terms, finding a solution of an optimal control problems over infinite-dimensional systems involves two main tasks: approximation of the system dynamics by means of classical schemes like the finite element method, and resolution of an optimization problem. Depending upon the problem and control requirements, these tasks can be performed in two different ways, discretizing and then optimizing or vice versa. Examples of the approach "optimize then discretize" can be studied in [3]. In particular, our work can be classified in the strategy "discretize then optimize" (see [1, 14]), as we first approximate the system dynamics using the finite element method and then solving a finite dimensional algebraic Riccati equation associated with the solution of the LQR problem. This last issue, the solution of large-scale algebraic Riccati equations arising in optimal controls problems after spatial discretization, has been extensively studied (see [4, 16]), providing reliable methods, which can be used under suitable conditions.

The aim of this work is to obtain optimal-order error estimates under the theory developed in [14] related to the approximation theory of optimal control problem of evolutionary systems over an infinite time interval. We consider the strongly damped wave equation representing the vibration of a string as the state equation, the vertical displacements and velocity as the state variables, and two kinds of controls: point and distributed. The goal of the problem is to compensate the vibrations arising of a set of initial conditions. In order to achieve this, we seek for a control signal represented in a feedback form from the state variables. By following the abstract theory stated in [14], a series of assumptions must be proved in order to obtain optimal convergence rates for the system output and control variables. These assumptions are connected with to stability and consistency properties of the approximated problems. The convergence rates relies on approximation properties of the control-free dynamics and the degree of unboundedness of the control operator.

There are many works in the control problem of the wave equation (see [10, 19]). Moreover, the LQR control strategy for second-order evolutionary systems has been studied and computationally implemented (see [12, 18]). Despite this, to the best of the authors knowledge, the mathematical analysis and computational validation of the optimal convergence rates for this control problem has not yet been performed.

The outline of this paper is as follows. In Sect. 2 we state the abstract optimal control problem and the conditions needed in order to prove existence and uniqueness of the exact solution for both cases. In Sect. 3 we deal with approximation issues: we state the approximated control problem for and prove the conditions that ensure optimal-order convergence rates. In Sect. 4 we present numerical results that validate the stabilization and the convergence rates obtained previously.

2 Abstract setting of the optimal control problem

We consider the strongly damped wave equation in the interval $\mathbf{I} = [0, L]$ with point and distributed controls acting as external sources; i.e., $x(\xi, t)$ represents a vertical displacement along a string that satisfies

$$\begin{cases} \dfrac{\partial^2 x(\xi,t)}{\partial t^2} - \dfrac{\partial^2 x(\xi,t)}{\partial \xi^2} - \rho \dfrac{\partial^3 x(\xi,t)}{\partial \xi^2 \partial t} = \bar{u}(\xi,t), & t > 0, \xi \in [0, L], \\ x(0,t) = x(L,t) = 0, & t > 0, \\ x(\xi,0) = f(\xi),\ x_t(\xi,0) = g(\xi), & \xi \in [0, L], \end{cases} \tag{1}$$

where ξ represents the spatial coordinate, t the time and ρ a damping coefficient. The external load $\bar{u}(\xi,t) = u_p(\xi,t) := \delta(\xi - \xi_0)u(t)$ denotes the point control in the point $\xi_0 \in (0, L)$ and $\bar{u}(\xi,t) = u_d(\xi,t)$ denotes a distributed control acting over $\mathbf{I}_c$ in the following way:

$$u_d(\xi,t) := \begin{cases} u(t) & \xi \in \mathbf{I}_c \\ 0 & \xi \in \mathbf{I}/\mathbf{I}_c. \end{cases}$$

We are interested in finding feedback control laws in the form

$$\bar{u}(\xi,t) = -Kx(\xi,t), \tag{2}$$

for the output regulation problem for the vibration of the string over an infinite time horizon, where K is a gain operator obtained from an algebraic Riccati equation that will be specified later.

In order to define the optimal control problems we start by writing our equation as a evolutionary system of first order. We set the operator

$$\mathcal{A} = -\frac{\partial^2(\cdot)}{\partial x^2}, \qquad \mathcal{D}(\mathcal{A}) = H^2(\mathbf{I}) \cap H_0^1(\mathbf{I}),$$

and the state vector

$$y = [x(\xi,t)\ \ \dot{x}(\xi,t)]^T,$$

where here and therein $\dot{x}$ stands for $\frac{\partial(\cdot)}{\partial t}$.

With this setting, we formally obtain the following first order state-space representation for (1):

$$\begin{cases} \dot{y}(\xi,t) = Ay + Bu, \\ y(\xi,0) = y_0, \end{cases} \tag{3}$$

with

$$A = \begin{bmatrix} O & I \\ -\mathcal{A} & -\rho\mathcal{A} \end{bmatrix}, \qquad B_p u_p = \begin{bmatrix} 0 \\ \delta(\xi - \xi_0)u(t) \end{bmatrix}, \qquad B_d u_d = \begin{bmatrix} 0 \\ \mathcal{I}_c u(t) \end{bmatrix} \tag{4}$$

where $B = B_p$ and $B = B_d$ denote the operators associated with point and distributed controls respectively, $\mathcal{I}_c : L^2(\mathbf{I}) \to L^2(\mathbf{I})$ is defined by $(\mathcal{I}_c v)(x) = \chi_c v(x)$ for all

$x \in \mathbf{I}$ and χ_c is the characteristic function of the subset $\mathbf{I}_c$. We consider both operators from $\mathbb{R}$ onto $[D(A^*)]'$.

For both control problems, let us consider a state-space $Y = H^1(\mathbf{I}) \times L^2(\mathbf{I})$, such that $\mathcal{D}(A) \subset Y$, a control space $U = \mathbb{R}$ and the cost functional

$$\mathcal{J}(y, u) = \frac{1}{2} \int_0^{\infty} \left\{ \|x(\xi, t)\|^2_{H^1(\mathbf{I})} + \|\dot{x}(\xi, t)\|^2_{L^2(\mathbf{I})} + |u(t)|^2 \right\} dt. \tag{5}$$

We state that the control laws are optimal in the sense that they minimize this functional.

Here and therein, for Z a function space, $z \in L^2(Z)$ stands for a function $z(\cdot, t) \in Z$ and $z(\xi, \cdot) \in L^2([0, +\infty[)$.

Note that the optimal control problems differs in the choice of the control operators B_p and B_d in (3).

It follows from the theory presented in Chap. 2 [14], that existence and uniqueness for the solution of these abstract control problems are guaranteed if:

(H.1) A is the infinitesimal generator of a strongly continuous, analytic semigroup, denoted by e^{At}, on Y.
(H.2) B_p and B_d are linear operators, such that $A^{-\gamma_p} B_p \in \mathcal{L}(U, Y)$ and $A^{-\gamma_d} B_d \in \mathcal{L}(U, Y)$, for some fixed constants γ_p, γ_d, respectively, with $\gamma_p, \gamma_d \in [0, 1)$.
(H.3) *Finite cost condition:* For each problem, given $y_0 \in Y$, there exists $\bar{u} \in L^2(0, \infty; U)$, such that $\mathcal{J}(\bar{u}, \bar{y}) < \infty$.

The condition (H.1) on Y follows from [7], as the damping operator can be identified with a fractional power α of the elastic operator, with $1/2 \leq \alpha \leq 1$; our case holds with $\alpha = 1$.

On the other hand, is clear that the condition (H.2) holds with $\gamma_d = 0$ for the distributed control problem. In the case of the point control problem, we first notice that the domain for the fractional powers of the operator A is given by the formula

$$D(A^\theta) = H^1_0(\mathbf{I}) \times H^{2\theta}_0(\mathbf{I}), \quad \forall \theta \leq 1/2.$$

The same characterization is valid for the adjoint operator. Then, as a consequence of $(A^{-\gamma_p} B_p u, v)_Y = u[A^{*-\gamma_p} v]_2(\xi_0)$, and since $A^{*-\gamma_p} v \in H^1_0(\mathbf{I}) \times H^{2\gamma_p}_0(\mathbf{I})$, we obtain that

$$[A^{*-\gamma_p} v]_2 \in H^{2\gamma_p}_0(\mathbf{I}) \subset C(\mathbf{I}),$$

for all $\gamma_p > 1/4$.

Finally, the finite cost condition is always satisfied; in fact, the damping factor allows to us to always take the control law $u \equiv 0$ such that $\mathcal{J}(0, \bar{y}) < \infty$. This is a direct consequence of the stability condition stated in Theorem 3.B.1, [14].

In general terms, the feedback control law given in (2) is related to the output by:

$$y(\xi, t) = e^{A_\Pi t} y_0, \qquad \bar{u}(\xi, t) = -B^* \Pi e^{A_\Pi t} y_0, \quad \forall t \geq 0,$$

where $A_\Pi = A - BB^* \Pi$ is the operator related with the closed-loop dynamics and $\Pi = \Pi^* \in \mathcal{L}(Y)$ is the unique nonnegative operator that satisfies the following algebraic Riccati equation (ARE):

$$(A^*\Pi x, y)_Y + (\Pi A x, y)_Y - (B^*\Pi x, B^*\Pi y)_U + (x, y)_Y = 0,$$

for all $(x, y) \in \mathcal{D}(A) \times \mathcal{D}(A)$, where $(\cdot, \cdot)$ denotes the inner product over the corresponding space.

3 Approximation results

Once that the abstract setting is given we construct an approximation scheme for the optimal control problems. Following the structure present in Chap. 4 [14], we start by selecting a finite-dimensional approximating subspace $\mathcal{V}_h \subset \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) = H_0^1(\mathbf{I})$, to be a piecewise linear finite element space. For this reason, we consider a family $\{\mathcal{T}_h\}$ of partitions of the interval $\mathbf{I}$:

$$\mathcal{T}_h : 0 = s_0 < s_1 < \cdots < s_n = L, \tag{6}$$

with mesh size

$$h := \max_{j=1,\ldots,n} (s_j - s_{j-1}).$$

Then, the subspace $\mathcal{V}_h$ can be written as follow:

$$\mathcal{V}_h := \left\{ v \in H_0^1(\mathbf{I}) : v|_{[s_{j-1} - s_j]} \in \mathbb{P}_1, \, j = 1, \ldots, n \right\}. \tag{7}$$

We let $V_h = \mathcal{V}_{h1} \times \mathcal{V}_{h2}$, where $\mathcal{V}_{h1}$ consists of the elements of $\mathcal{V}_h$ equipped with the $H^1(\mathbf{I})$ seminorm and $\mathcal{V}_{h2}$ consists of the elements of $\mathcal{V}_h$ equipped with the $L^2(\mathbf{I})$ norm.

We denote by $\mathcal{P}_h$ the orthogonal projection from $L^2(\mathbf{I}) \times L^2(\mathbf{I})$ onto V_h by

$$\mathcal{P}_h = \begin{bmatrix} \pi_h & 0 \\ 0 & \pi_h \end{bmatrix} \tag{8}$$

where π_h represents the orthogonal projection from $L^2(\mathbf{I})$ onto $\mathcal{V}_h$. The subspace $\mathcal{V}_h$ satisfy the approximation property

$$\|\pi_h x - x\|_{H^l(\mathbf{I})} \leq C h^{s-l} \|x\|_{H^s(\mathbf{I})}, \quad x \in H^s(\mathbf{I}) \cap H_0^1(\mathbf{I}), \tag{9}$$

with $0 \leq l \leq s \leq 2$.

Then, we can write the Galerkin approximation of the operator A on V_h as

$$A_h = \begin{bmatrix} O & \pi_h \\ -\mathcal{A}_h & -\rho \mathcal{A}_h \end{bmatrix} : V_h \to V_h \tag{10}$$

where $\mathcal{A}_h$ is the Galerkin approximation of the operator $\mathcal{A}$, i.e., $\mathcal{A}_h = \pi_h \mathcal{A} : \mathcal{V}_h \to \mathcal{V}_h$ such that $(\mathcal{A}_h x_h, v_h)_{L^2(\mathbf{I})} = (\mathcal{A} x_h, v_h)_{L^2(\mathbf{I})}, \; x_h, v_h \in \mathcal{V}_h$. Using the same ideas, we can write the approximations B_{ph} of B_p and B_{dh} of B_d respectively, in (4), by

$$B_{ph}u := \mathcal{P}_h B_p u = \begin{bmatrix} 0 \\ \mathcal{B}_h u(t) \end{bmatrix} : \mathbb{R} \to V_h, \qquad (\mathcal{B}_h u, v_h)_{L^2(\mathbf{I})} = v_h(\xi_0)u, \quad (11)$$

$$B_{dh}u := \mathcal{P}_h B_d u = \begin{bmatrix} 0 \\ \pi_h \mathcal{I}_c u(t) \end{bmatrix} = \begin{bmatrix} 0 \\ \mathcal{I}_c u(t) \end{bmatrix} : \mathbb{R} \to V_h. \quad (12)$$

It is easy to verify that the adjoint operators of A_h, B_{ph} and B_{dh} in (10), (11) and (12) respectively, are given by:

$$A_h^* = \begin{bmatrix} O & -\pi_h \\ \mathcal{A}_h & \rho \mathcal{A}_h \end{bmatrix} : V_h \to V_h, \quad (13)$$

$$B_{ph}^* v_h = v_{h2}(\xi_0), \qquad v_h = [v_{h1}\ v_{h2}]^T, \quad (14)$$

$$B_{dh}^* v_h = \begin{bmatrix} 0 \\ (v_{h2}, \chi_c)_{L^2(\mathbf{I})} \end{bmatrix}, \qquad v_h = [v_{h1}\ v_{h2}]^T. \quad (15)$$

Now we seek for an approximated solution $(\bar{y}_h, \bar{u}_h)$ of our optimal control problems:

$$\begin{aligned} &\inf \mathcal{J}(y_h, u_h) = \frac{1}{2} \int_0^\infty \left\{ \|y_h(\xi, t)\|_Y^2 + |u_h(t)|^2 \right\} dt \\ &\text{s.t.} \quad \dot{y}_h = A_h y_h + B_h u_h, \\ &\qquad\quad y_h(0) = \mathcal{P}_h y_0. \end{aligned}$$

The approximating dynamics $\dot{y}_h = A_h y_h + B_h u_h$ are given, via (10)–(12), by

$$\begin{cases} (\ddot{x}_h, v_h) - (\mathcal{A}_h x_h, v_h) - \rho(\mathcal{A}_h \dot{x}_h, v_h) = (B_h u_h, v_h) & \forall v_h \in \mathcal{V}_h, \\ (\mathcal{A}_h x_h, v_h) = -(x_h', v_h') & \forall v_h \in \mathcal{V}_h, \\ (x_h(0), v_h) = (f, v_h), \qquad (\dot{x}_h(0), v_h) = (g, v_h) & \forall v_h \in \mathcal{V}_h, \end{cases} \quad (16)$$

with $y_h = [x_h \quad \dot{x}_h]$, and all the interior products are taken in $L^2(\mathbf{I})$. The optimal feedback control law for the approximated problem is

$$\bar{u}_h(t, \mathcal{P}_h y_0) = -B_h^* \Pi_h e^{A_{\Pi_h} t} \mathcal{P}_h y_0$$

and Π_h is the unique nonnegative, self-adjoint solution of the following algebraic Riccati equation (ARE$_h$):

$$(A_h^* \Pi_h \phi_h, v_h)_Y + (\phi_h, A_h^* \Pi_h v_h)_Y - (B_h^* \Pi_h \phi_h, B_h^* \Pi_h v_h)_U + (\phi_h, v_h)_Y = 0 \quad (17)$$

$\forall (\phi_h, v_h) \in V_h \times V_h$.

Our goal is to obtain optimal convergence rates in both cases, point and distributed. To get it, we follow the abstract framework of optimal control theory for partial differential equations, as stated in Chap. 4 of [14]. We will begin with the point optimal control problem for which we need to prove the assumptions stated in Theorem 4.1.4.1 in [14], that in this case turns to be:

(A.1P) A_h is the infinitesimal generator of a uniformly analytic semigroup on V_h.

(A.2P)

$$\|A^{-1}\mathcal{P}_h - A_h^{-1}\mathcal{P}_h\|_{\mathcal{L}(H_0^1(\mathbf{I})\times L^2(\mathbf{I}))} \leq Ch.$$

(A.3P)

$$|B_p^* x_h| \leq Ch^{-2\gamma_p}\|x_h\|_{H_0^1(\mathbf{I})\times L^2(\mathbf{I})}, \quad \forall x_h \in V_h.$$

(A.4P)

$$|B_p^*(\mathcal{P}_h - I)x| \leq Ch^{1-2\gamma_p}\|x\|_{D(A^*)}, \quad \forall x \in D(A^*).$$

(A.5P)

$$|B_p^*\mathcal{P}_h x| \leq C\|(A^*)^{\gamma_p}x\|_Y, \quad \forall x \in D((A^*)^{\gamma_p}).$$

Note that (A.2P) is a variant from the original assumption that allows us to recover the same convergence rates, where the initial condition is replaced by his projection onto V_h. Also notice that (A.5) in [14] is omitted because $B_{ph} = \mathcal{P}_h B_p$.

Lemma 1 *For the point control problem presented*, (A.1P)–(A.5P) *holds.*

Proof Each proof will be given separately.

(A.1P) The fact that A_h is the infinitesimal generator of a uniformly analytic semigroup on $\mathcal{V}_h$, follows from the application of the arguments presented in [7], with $\alpha = 1$, to the finite-dimensional operator A_h.

(A.2P) From definition of $\mathcal{P}_h$, noting that

$$A^{-1} = \begin{bmatrix} -\rho I & -\mathcal{A}^{-1} \\ I & 0 \end{bmatrix}, \qquad A_h^{-1} = \begin{bmatrix} -\rho\pi_h & -\mathcal{A}_h^{-1} \\ \pi_h & 0 \end{bmatrix},$$

and that the interior product in $H_0^1(\mathbf{I}) \times L^2(\mathbf{I})$ is defined as

$$\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}\right)_{H_0^1(\mathbf{I})\times L^2(\mathbf{I})} = (x_1, y_1)_{H^1(\mathbf{I})} + (x_2, y_2)_{L^2(\mathbf{I})},$$

we obtain

$$\begin{aligned} \|(A^{-1}\mathcal{P}_h - A_h^{-1}\mathcal{P}_h)x\|_{H_0^1(\mathbf{I})\times L^2(\mathbf{I})} &\leq \|\mathcal{A}^{-1}\pi_h x_2 - \mathcal{A}_h^{-1}\pi_h x_2\|_{H^1(\mathbf{I})} \\ &\leq Ch\|x_2\|_{L^2(\mathbf{I})} \end{aligned}$$

where the last inequality is obtained using standard finite element approximation (see, for example [9]).

(A.3P) Because of the Sobolev embedding $H^m(\mathbf{I}) \hookrightarrow L^\infty(\mathbf{I})$ for all $m > 1/2$ and a inverse approximation property (see Chap. 3, [9]), we have that

$$
\begin{aligned}
|B_p^* x_h| = |x_{h2}(\xi_0)| &\le C\|x_{h2}\|_{H^{1/2+\epsilon}(\mathbf{I})} \\
&\le Ch^{-1/2-\epsilon}\|x_{h2}\|_{L^2(\mathbf{I})} \\
&\le Ch^{-2\gamma_p}\|x_h\|_Y.
\end{aligned}
$$

The last inequality is justified by the fact that in our case $\gamma_p > 1/4$.

(A.4P) By means of the Sobolev embedding used in the proof of (A.3P) and the approximation property (9),

$$
\begin{aligned}
|B_p^*(\mathcal{P}_h - I)x| = |\pi_h x_2(\xi_0) - x_2(\xi_0)| &\le C\|\pi_h x_2 - x_2\|_{H^{1/2+\epsilon}(\mathbf{I})} \\
&\le Ch^{1-1/2-\epsilon}\|x_2\|_{H^1(\mathbf{I})} \\
&\le Ch^{1-1/2-\epsilon}\|x_2\|_{D(A^*)} \\
&\le Ch^{1-2\gamma_p}\|x_2\|_{D(A^*)}.
\end{aligned}
$$

(A.5P) Using the same Sobolev embedding,

$$
\begin{aligned}
|B^*\mathcal{P}_h x| = |x_{h2}(\xi_0)| &\le C\|x_{h2}\|_{H^{1/2+\epsilon}(\mathbf{I})} \\
&\le C\|x_h\|_{D((A^*)^{\gamma_p})}.
\end{aligned}
$$

The last inequality is a consequence of Theorem 1.1 in [8]: $D((A^*)^{\gamma_p} \subset H_0^1(\mathbf{I}) \times H^{2\gamma_p}(\mathbf{I})$ and $2\gamma_p = 2(1/4+\epsilon) > 1/2+\epsilon$. □

Now we state our first main result, which gives optimal convergence rates for the point control problem.

Theorem 1 *There exists $h_0 > 0$ such that for all $h < h_0$ (ARE_h) in (17), with $B = B_p$, admits a unique, nonnegative, self-adjoint solution Π_h. Moreover, there exists $\omega_0 > 0$, such that for any $\epsilon > 0$, $t > 0$, the following convergence rates are obtained*:

$$
|\bar{u}_p(\cdot,t) - \bar{u}_{ph}(\cdot,t)| \le C\frac{e^{-\omega_0 t}}{t^{1/2}}h^{1/2-\epsilon}\|\mathcal{P}_h y_0\|_{H_0^1(\mathbf{I})\times L^2(\mathbf{I})} \tag{18}
$$

$$
\|\bar{y}(\cdot,t) - \bar{y}_h(\cdot,t)\|_{H_0^1(\mathbf{I})\times L^2(\mathbf{I})} \le C\frac{e^{-\omega_0 t}}{t^{1-\epsilon}}h^{1/2-\epsilon}\|\mathcal{P}_h y_0\|_{H_0^1(\mathbf{I})\times L^2(\mathbf{I})}. \tag{19}
$$

Proof The existence and uniqueness of the solution of the abstract control problem follows from (H.1)–(H.3). Using Lemma 1 and due to the compactness of the operator $B_p^* A^{*^{-1}} : (L^2(\mathbf{I}))^2 \longrightarrow \mathbb{R}$ (because the injection of H^1 in L^2 is compact), the existence of h_0 that verifies the first part of the result follows directly from Theorem 4.1.4.1 in [4].

On the other hand, since $B_{ph} = \mathcal{P}_h B_p$, the assumptions (A7)–(A9) in Chap. 4 of [14], are automatically satisfied with $r_0 = r_1 = 1 - 2\gamma_p$, as (A.3P) and (A.4P) are valid for $2\gamma_p = 1/2+\epsilon$. Then, applying Theorem 4.6.2.2 in [14], we conclude the estimates (18) and (19). □

We study the distributed control problem. The assumptions for this case reads as:

(A.1D) A_h is the infinitesimal generator of a uniformly analytic semigroup on V_h.

(A.2D)

$$\|A^{-1}\mathcal{P}_h - A_h^{-1}\mathcal{P}_h\|_{\mathcal{L}(H_0^1(\mathbf{I})\times L^2(\mathbf{I}))} \leq Ch.$$

(A.3D)

$$|B_d^* x_h| \leq C\|x_h\|_{H_0^1(\mathbf{I})\times L^2(\mathbf{I})}, \quad \forall x_h \in V_h.$$

(A.4D)

$$|B_d^*(\mathcal{P}_h - I)x| \leq Ch\|x\|_{D(A^*)}, \quad \forall x \in D(A^*).$$

(A.5D)

$$|B_d^*\mathcal{P}_h x| \leq C\|x\|_{H_0^1(\mathbf{I})\times L^2(\mathbf{I})}, \quad \forall x \in H_0^1(\mathbf{I}) \times L^2(\mathbf{I}).$$

As in the point problem, (A.5) in [14] is omitted because $B_{dh} = \mathcal{P}_h B_d$.

Lemma 2 *For the distributed control problem*, (A.1D)–(A.5D) *holds*.

Proof Each proof will be given separately.

(A.1D) Since this property is related to the uniform analyticity of the semigroup generated by A_h over V_h, the proof is identical to (A.1P).

(A.2D) Since this property is related to an approximation property of the control-free dynamics A, the proof is identical to (A.2P).

(A.3D) Using Cauchy–Schwarz inequality we have

$$|B^* x_h| = \int_{\mathbf{I}_c} |x_{h2}|d\xi \leq |\mathbf{I}_c|^{\frac{1}{2}}\|x_{h2}\|_{L^2(\mathbf{I}_c)} \leq |\mathbf{I}_c|^{\frac{1}{2}}\|x_h\|_{H_0^1(\mathbf{I})\times L^2(\mathbf{I})}.$$

(A.4D) Using Cauchy–Schwartz inequality and property (9)

$$|B^*(\mathcal{P}_h - I)x| = \int_{\mathbf{I}_c} |\pi_h x_2 - x_2|d\xi \leq |\mathbf{I}_c|^{\frac{1}{2}}\|\pi_h x_2 - x_2\|_{L^2(\mathbf{I})} \leq Ch\|x_2\|_{D(A^*)}.$$

(A.5D) Using Cauchy–Schwarz inequality and that π_h is a continuous application over $L^2(\mathbf{I})$, there holds

$$|B_d^*\mathcal{P}_h x| = \int_{\mathbf{I}_c} |x_{h2}|d\xi \leq |\mathbf{I}_c|^{\frac{1}{2}}\|x_{h2}\|_{L^2(\mathbf{I})} \leq C\|x\|_{H_0^1(\mathbf{I})\times L^2(\mathbf{I})}.$$ □

Now we state our second main result, which gives optimal convergence rates for the distributed control problem.

Theorem 2 *There exists $h_0 > 0$ such that for all $h < h_0$ (ARE_h) in (17), with $B = B_p$, admits a unique, nonnegative, self-adjoint solution Π_h. Moreover, there exists $\omega_0 > 0$, such that for any $\epsilon > 0$, $t > 0$, the following convergence rates are obtained:*

$$|\bar{u}_p(\cdot,t) - \bar{u}_{ph}(\cdot,t)| \le C\frac{e^{-\omega_0 t}}{t^{-\epsilon}}h^{1-\epsilon}\|\mathcal{P}_h y_0\|_{H_0^1(\mathbf{I})\times L^2(\mathbf{I})}$$

$$\|\bar{y}(\cdot,t) - \bar{y}_h(\cdot,t)\|_{H_0^1(\mathbf{I})\times L^2(\mathbf{I})} \le C\frac{e^{-\omega_0 t}}{t^{1-\epsilon}}h^{1-\epsilon}\|\mathcal{P}_h y_0\|_{H_0^1(\mathbf{I})\times L^2(\mathbf{I})}.$$

Proof The proof is essentially the same as in Theorem 1. □

4 Computational implementation and numerical examples

In this section we give a computational solution of the above mentioned problem in order to exhibit the optimal convergence rates obtained theoretically.

We consider an uniform partition of the interval $\mathbf{I}$, $\mathcal{T}_h$ as in (6), and the finite-dimensional space of piecewise linear and continuous functions over $\mathbf{I}$ that vanishes in $\xi = 0$ and $\xi = L$, i.e. $\mathcal{V}_h$, defined in (7). We seek a solution of (16) assuming a Galerkin approximation of the form

$$x^N(\xi,t) = \sum_{j=1}^{N} c_j(t)\varphi_j(\xi),$$

where $\{\varphi_j\}_{j=1}^N$ denotes a basis of $\mathcal{V}_h$ and $N = \dim(\mathcal{V}_h)$.

Now, replacing this expression in (16) we obtain a second-order system of differential equations of the form:

$$M^N\ddot{c}(t) + D^N\dot{c}(t) + K^N c(t) = B_0^N u(t)$$

for $c(t) = [c_1(t), c_2(t), \ldots, c_N(t)]$, where the mass matrix M^N, the damping matrix D^N and the stiffness matrix K^N are given by:

$$M_{ij}^N = [(\varphi_i,\varphi_j)_{L^2(\mathbf{I})}],$$

$$D_{ij}^N = [\rho(\varphi_i',\varphi_j')_{L^2(\mathbf{I})}],$$

$$K_{ij}^N = [(\varphi_i',\varphi_j')_{L^2(\mathbf{I})}].$$

The actuator influence vectors B_p^N and B_d^N for both point and distributed cases respectively, are given by:

$$B_{p_i}^N = [\varphi_i(\xi_0)], \qquad B_{d_i}^N = [(\varphi_i, 1)_{L^2(\mathbf{I}_c)}].$$

The initial conditions for this second-order problem are obtained taking the Galerkin approximation of the initial conditions of the continuous problem,

$$(x^N(0),\varphi_j)_{L^2(\mathbf{I})} = (f,\varphi_j)_{L^2(\mathbf{I})},$$

$$(\dot{x}^N(0),\varphi_j)_{L^2(\mathbf{I})} = (g,\varphi_j)_{L^2(\mathbf{I})}.$$

Defining the vector state in the same way as we have done in the abstract problem, i.e. $\eta = [c(t), \dot{c}(t)]^T$, we formally obtain a classical first-order state-space representation form for the system dynamics:

$$\dot{\eta} = A^N \eta + B^N u, \qquad \eta(0) = \eta_0$$

where

$$A^N = \begin{bmatrix} 0 & I \\ -(M^N)^{-1} K^N & -\rho (M^N)^{-1} K^N \end{bmatrix}$$

and B^N changes accordingly the type of control:

$$B^N = \begin{bmatrix} 0 \\ -(M^N)^{-1} B_p^N \end{bmatrix}, \qquad B^N = \begin{bmatrix} 0 \\ -(M^N)^{-1} B_d^N \end{bmatrix}$$

for the point and distributed control problem respectively.

As our goal is to compute a solution for our approximated control problem we must solve now an algebraic Riccati equation for Π^N:

$$A^N \Pi^N + \Pi^N A^N - \Pi^N B^N (B^N)^T \Pi^N + Q^N = 0$$

where Q^N reflects the spatial norm taken in (16),

$$Q^N = \begin{bmatrix} M^N + K^N & 0 \\ 0 & M^N \end{bmatrix},$$

for the point and distributed control problem in $H_0^1(\mathbf{I}) \times L^2(\mathbf{I})$. Finally, the discrete control law is given by $u_h = -(B^N)^T \Pi^N \eta$.

We present now numerical experiments which are consistent with the above developed theoretical framework. We performed our simulations in `MatLab`, determining the suboptimal gains with the command `lqr`, and then advancing in time with a Runge–Kutta 4th order solver. In absence of an exact solution, all calculations related to error estimates have been obtained with respect to the approximated system output and input, y_{app} and u_{app} respectively, obtained with 1300 nodes.

We consider two different cases: first we solve a control problem with a point actuator at $\xi = 0.4$ and in a second example, a distributed actuator over $\mathbf{I}_c = [0.4, 0.6]$, showing stabilization and convergence rates for both the system output and the control signal at different instants $t_1 = 0.2$, $t_2 = 0.4$ and $t_3 = 0.6$; in both cases the damping factor ρ is taken equal to 0.005.

4.1 Stabilization

Figures 1 and 2 show stabilization of the state variables in both control problems, point and distributed respectively.

It can be observed, as it was theoretically stated in the finite cost condition, that due to the damping term, the system is stable; then in absence of control an exponentially bounded decay is observed for both the vertical displacement and velocity.

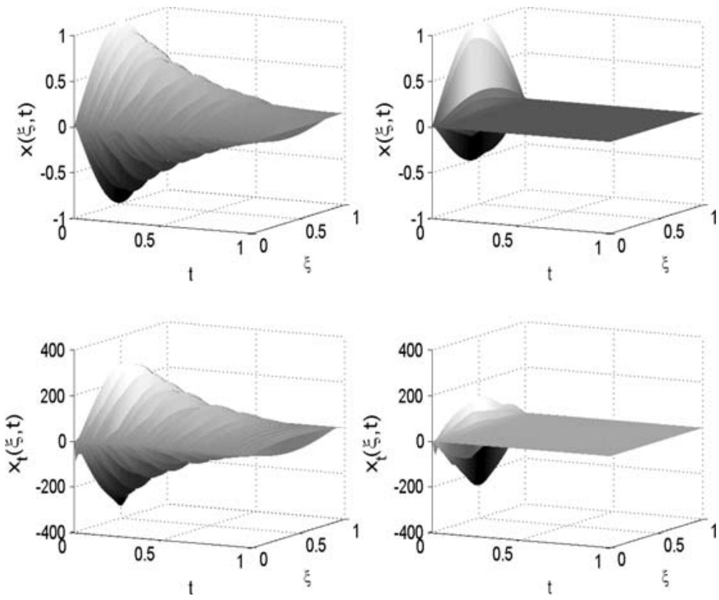

Fig. 1 1. Uncontrolled vertical displacement (*up-left*). 2. Controlled vertical displacement with point control at $\xi = 0.4$ (*up-right*). 3. Uncontrolled vertical velocity (*down-left*). 4. Controlled vertical velocity with point control at $\xi = 0.4$ (*down-right*)

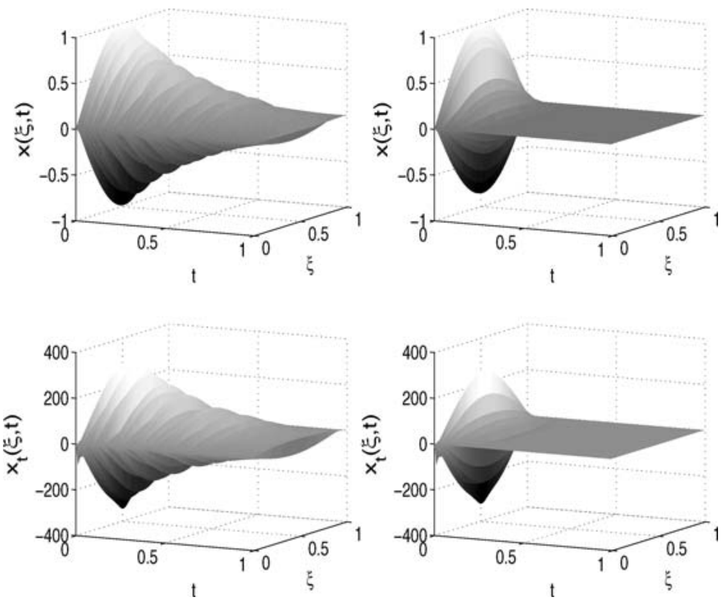

Fig. 2 1. Uncontrolled vertical displacement (*up-left*). 2. Controlled vertical displacement with distributed control along $\mathbf{I}_c = [0.4,\ 0.6]$ (*up-right*). 3. Uncontrolled vertical velocity (*down-left*). 4. Controlled vertical velocity with distributed control along $\mathbf{I}_c = [0.4,\ 0.6]$ (*down-right*)

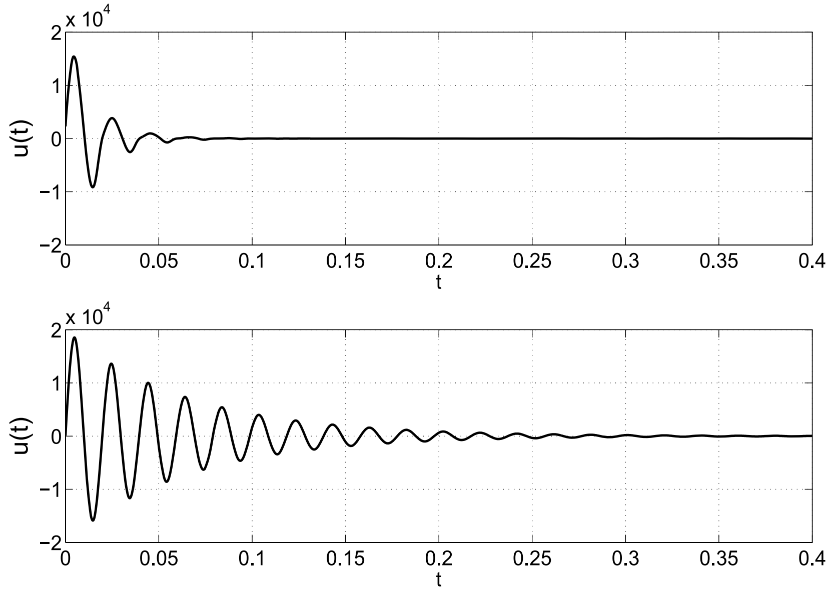

Fig. 3 1. Control signal for the point problem (*up*). 2. Control signal for the distributed problem (*down*)

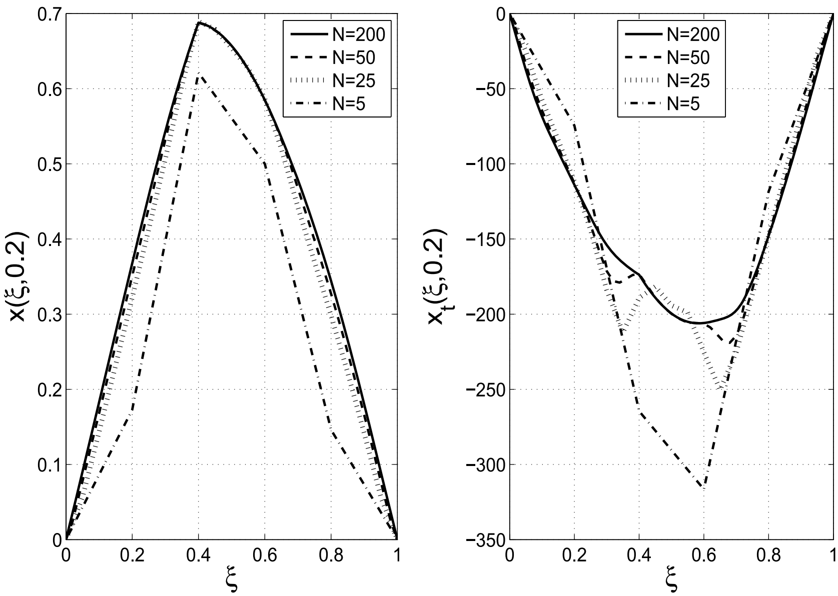

Fig. 4 1. Vertical displacement at $t = 0.2$ for different numbers of nodes in the point control problem (*left*). 2. Vertical velocity at $t = 0.2$ for different numbers of nodes in the point control problem (*right*)

This behavior is dramatically accelerated with the presence of a control acting over the system. However, the speed of the stabilization is directly related with the power

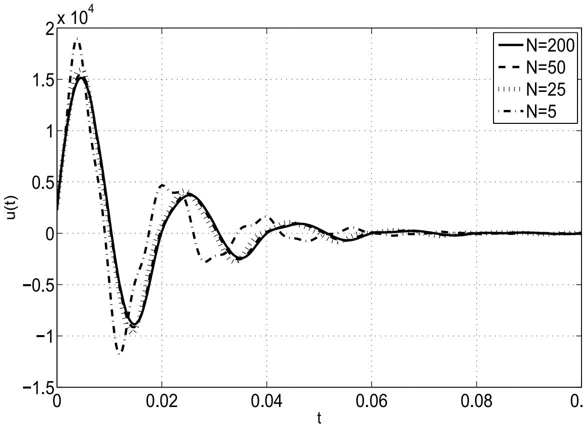

Fig. 5 Control signal evolution for different numbers of nodes for the point control problem

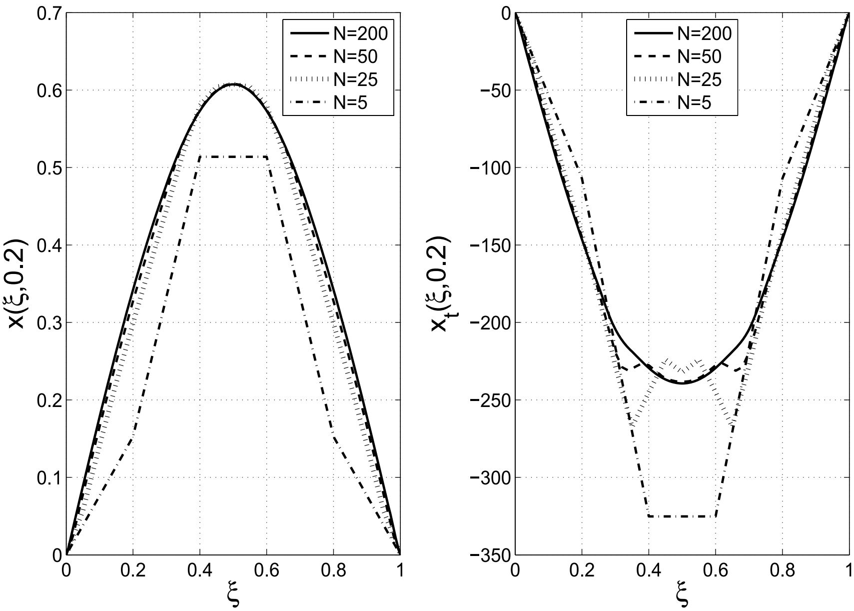

Fig. 6 1. Vertical displacement at $t = 0.2$ for different numbers of nodes in the distributed control problem (*left*). 2. Vertical velocity at $t = 0.2$ for different numbers of nodes in the distributed control problem (*right*)

of the control signal (see Fig. 3). This trade-off can be managed introducing weight factors for the input and output spatial norms present in (5).

4.2 Convergence rates

Basic computational validation for convergence in both problems is shown in Figs. 4–7. It can be seen that, in both point and distributed control problems, convergence for the states variables at a fixed instant is achieved by augmenting the number of nodes (see Figs. 4 and 6). Convergence for control signals, in space and consequently

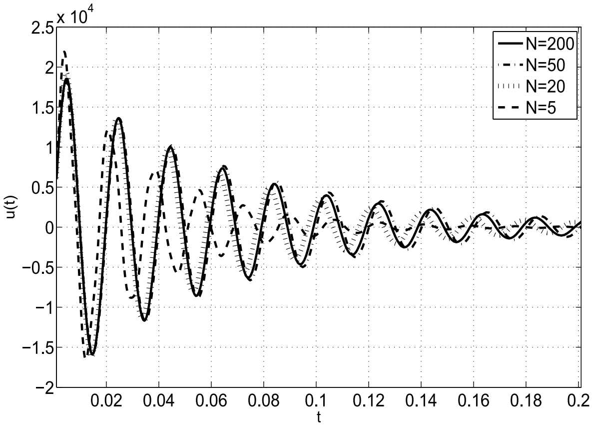

Fig. 7 Control signal evolution for different numbers of nodes for the distributed control problem

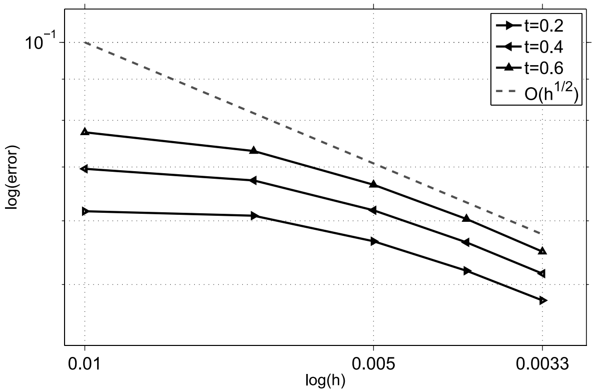

Fig. 8 $H^1(\mathbf{I})$ norms of the error for vertical displacement x at different instants (point control problem)

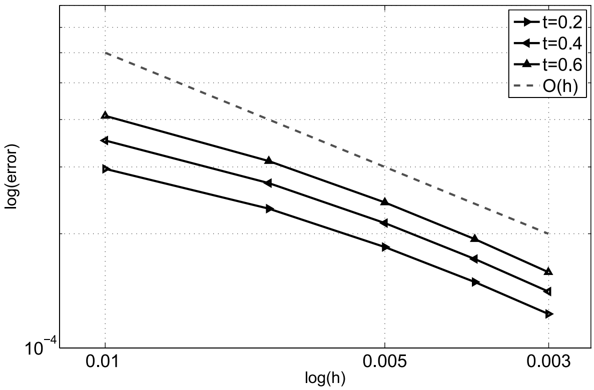

Fig. 9 $L^2(\mathbf{I})$ norms of the error for the vertical velocity $\dot{x}$ at different instants (point control problem)

in time is also observed in Figs. 5 and 7. Note that, we also include coarse meshes such that the convergence behaviour can be seen.

Comparison between theoretically predicted and computationally obtained convergence rates can be observed in Figs. 8–13.

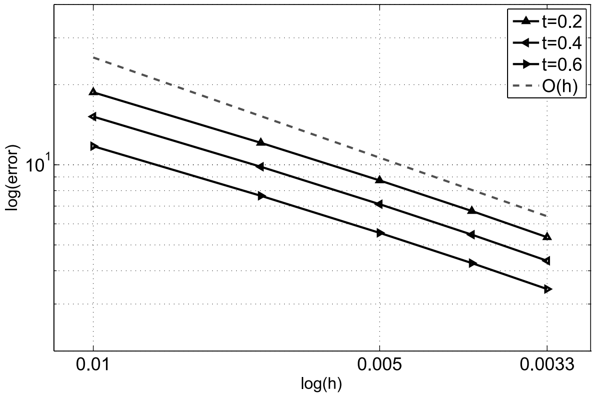

Fig. 10 $L^2(\mathbf{I})$ norms of the error for the input u at different instants (point control problem)

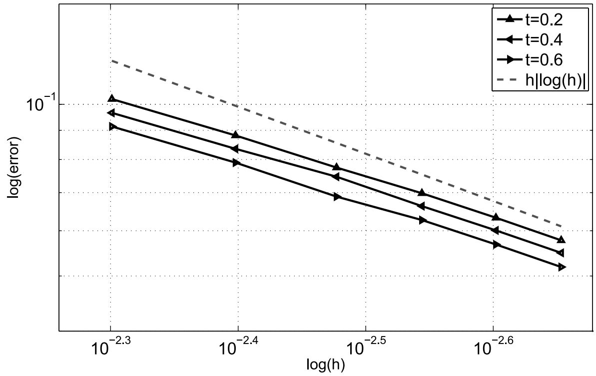

Fig. 11 $H^1(\mathbf{I})$ norms of the error for vertical displacement x at different instants (distributed control problem)

The vertical displacement x converges with order $\mathcal{O}(h^{1/2})$, which is consistent with theoretically predicted order $\mathcal{O}(h^{1/2-\epsilon})$ (see Fig. 8). On the other hand, according to Fig. 9, the $L^2(\mathbf{I})$ norm of the vertical velocity converges faster with an experimental order similar to $\mathcal{O}(h)$. This is not contradictory with Theorem 1, as the norm of y is taken in $H^1(\mathbf{I}) \times L^2(\mathbf{I})$, the convergence is governed by the slower term, which in this case turns to be the theoretically predicted order $\mathcal{O}(h^{1/2-\epsilon})$.

In Fig. 10, it can be observed that the control u converges faster than the order stated in (18); indeed, it converges with the same order than the vertical velocity. This is caused by the fact that the control is obtained from a feedback of both the vertical displacement and vertical speed. Due to a scaling issue (see Fig. 1), the contribution for the feedback of the vertical displacement is negligible in comparison with the vertical speed. Then, even if theoretical derivation of the convergence rate for the control both contributions are considered in a similar manner, this is not experimentally observed.

Figures 11–13 show convergence rates for the distributed control problem. By similar arguments as in [15], the results stated in Theorem 2 can be replaced with orders $\mathcal{O}(h|log(h)|)$, independent of ϵ. This order is experimentally validated for both system output and input, which converges slightly faster with order $O(h)$ by the same explanation than in the point control case.

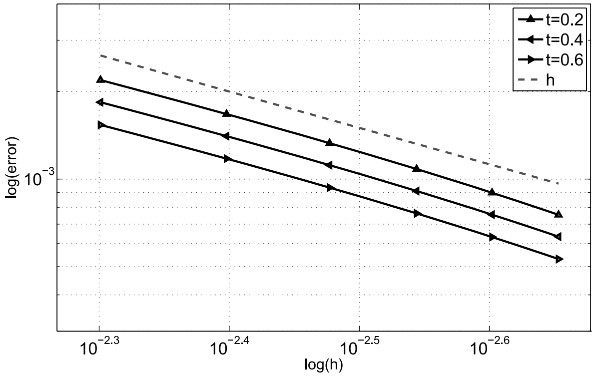

Fig. 12 $L^2(\mathbf{I})$ norms of the error for the vertical velocity $\dot{x}$ at different instants (distributed control problem)

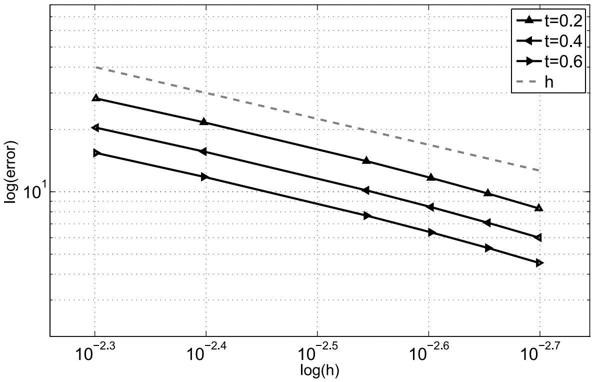

Fig. 13 $L^2(\mathbf{I})$ norms of the error for the input u at different instants (distributed control problem)

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