# FINITE ELEMENT DISCRETIZATION OF WEIGHTED $p$-LAPLACE PROBLEMS*

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**Abstract.** We study the finite element approximation of problems involving the weighted $p$-Laplacian for $p \in (1,\infty)$ and weights belonging to the Muckenhoupt class $A_1$. In particular, we consider an equation and an obstacle problem for the weighted $p$-Laplacian and derive error estimates in both cases. The analysis is based on the language of weighted Orlicz and Orlicz–Sobolev spaces.

**Key words.** nonlinear elliptic equations, weighted $p$-Laplacian, obstacle problem, Muckenhoupt weights, finite element discretizations, quasi norms, a priori error estimates.

**AMS subject classifications.** 35J60, 35J70, 65N15, 65N30.

## 1. Introduction.

Let $d \in \mathbb{N}$ with $d \geq 2$, and let $\Omega$ be a bounded polytope in $\mathbb{R}^d$ with Lipschitz boundary $\partial\Omega$. The aim of this paper is to study finite element methods for approximating the unconstrained and constrained minimization of the following convex energy functional

$$\mathcal{J}(v) \coloneqq \int_\Omega \omega(x)\varphi(|\nabla v(x)|)\mathrm{d}x - \int_\Omega \omega(x) f(x) v(x)\mathrm{d}x, \tag{1.1}$$

over an appropriate function class that satisfies homogeneous Dirichlet boundary conditions in particular. Here $\varphi$ is a suitable convex function, $\omega$ is a weight belonging to a Muckenhoupt class, and $f$ is a given forcing term; see Section 2 for precise assumptions.

The idea behind the study of minimization problems for energies such as (1.1) is that in the resulting differential operator, i.e.,

$$D\mathcal{J}(v) = -\operatorname{div}(\boldsymbol{\mathcal{A}}(x,\nabla v)),$$

the vector field $\boldsymbol{\mathcal{A}}$ may have a nonstandard growth (through the function $\varphi$) and also a non-translation invariant property (due to the dependence on the point $x$). The only simplifying assumption we make is that we consider each of these phenomena separately in the sense that

$$\boldsymbol{\mathcal{A}}(x,\boldsymbol{\zeta}) = \omega(x)\mathbf{A}(\boldsymbol{\zeta}), \qquad \mathbf{A}(\boldsymbol{\zeta}) = \varphi'(|\boldsymbol{\zeta}|)\frac{\boldsymbol{\zeta}}{|\boldsymbol{\zeta}|}$$

for a.e. $x \in \Omega$ and all $\boldsymbol{\zeta} \in \mathbb{R}^d$. The prototypical example of the type of problem we have in mind is the so-called weighted $p$-Laplacian: for $p \in (1,\infty)$ and $\kappa \geq 0$, we have

$$\mathbf{A}(\boldsymbol{\zeta}) = (\kappa + |\boldsymbol{\zeta}|)^{p-2}\boldsymbol{\zeta} \qquad \forall \boldsymbol{\zeta} \in \mathbb{R}^d. \tag{1.2}$$

The analysis and numerical approximation of quasilinear problems in general and the $p$-Laplacian in particular have a rich history, and any attempt to give a complete

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bibliographical account is doomed to failure. We mention here only the classical references [4] and [17]. In particular, in [17], the authors introduce the language of Orlicz functions to perform an error analysis. For the approximation of obstacle problems with the $p$-Laplacian as operator, we may refer, for instance, to [2, 11].

On the other hand, the study of elliptic problems with nonstandard growth and inhomogeneity has received much attention in recent years, as this seems not only to adopt the difficulties of certain scenarios, but also to unify certain particular structures; see [3, 31, 13]. It is therefore only natural to investigate the approximation of solutions to such problems. This work can therefore be seen as a step towards the numerical approximation of elliptic problems in Orlicz–Musielak spaces.

We structure our presentation as follows. Section 2 introduces the notation and gives an overview of the main ingredients we will use. The goal is to properly describe the functional framework we will use: weighted Orlicz and Orlicz–Sobolev spaces. This will be done in Section 2.5. The unconstrained minimization of (1.1) and its numerical approximation will be discussed in Section 3. The analysis hinges on the properties of the Scott-Zhang operator on weighted Orlicz spaces. Finally, an obstacle problem related to (1.1) and its numerical approximation is studied in Section 4. In this case, the analysis relies on a positivity preserving interpolant and its properties. This operator is analyzed in Section 4.3.

**2. Notation and preliminary remarks.** The first notation we introduce is the relation $A \lesssim B$. This means that $A \leq CB$ for a nonessential constant $C > 0$, which can change at each occurrence. The relation $A \gtrsim B$ means $B \lesssim A$, and $A \simeq B$ is the short form for $A \lesssim B \lesssim A$. If it is necessary to explicitly mention a constant $C$, we assume that $C > 0$ and that the value can change at each occurrence.

Let $E \subset \mathbb{R}^d$ be a measurable set. We denote by $|E|$ the $d$-dimensional Lebesgue measure of the set $E$. If $0 < |E| < \infty$ and $f \in L^1_{\text{loc}}(\mathbb{R}^d)$, we define the mean value of the function $f$ over the set $E$ by

$$\langle f \rangle_E \coloneqq \fint_E f(x)\,\mathrm{d}x = \frac{1}{|E|}\int_E f(x)\,\mathrm{d}x.$$

Here and in the following, the relation $A \coloneqq B$ indicates that $A$ is *by definition* equal to $B$. The notation $B =: A$ means $A \coloneqq B$.

**2.1. N-functions, complementary functions, and shifted N-functions.** We say that $\Phi : [0,\infty) \to [0,\infty)$ is an $N$-function if $\Phi$ is differentiable and its derivative $\Phi'$ satisfies the following properties: $\Phi'(0) = 0$, $\Phi'(s) > 0$ for $s > 0$, $\Phi'$ is right-continuous at any point $s \geq 0$, $\Phi'$ is nondecreasing on $[0,\infty)$, and $\lim_{s\to\infty} \Phi'(s) = \infty$ [39, Definition 4.2.1]. We note that every $N$-function is convex [39, Lemma 4.2.2], [33, page 7]. We say that an $N$-function $\Phi$ satisfies the $\Delta_2$-condition (we shall write $\Phi \in \Delta_2$) if there exists a positive constant $K$ such that $\Phi(2t) \leq K\Phi(t)$ for all $t \geq 0$. The smallest of these constants is denoted by $\Delta_2(\Phi)$ [39, Definition 4.4.1], [33, page 23]. Since $\Phi(t) \leq \Phi(2t)$, the $\Delta_2$-condition guarantees that $\Phi(t) \simeq \Phi(2t)$ for all $t \geq 0$. More generally, if $\Delta_2(\Phi) < \infty$ and $a > 1$ is fixed, then $\Phi(t) \simeq \Phi(at)$ for all $t \geq 0$ [39, Exercise 4.5.5], [33, inequality (4.2)]. Finally, we mention that if $\Phi$ is an $N$-function and $\Delta_2(\Phi) < \infty$, then $\Phi(t) \simeq \Phi'(t)\,t$ uniformly in $t \geq 0$ [34, Proposition 7(2.6b)].

**2.1.1. Complementary functions.** Let $\Phi$ be an $N$-function. We set [39, Definition 4.3.1]

$$(\Phi')^{-1} : [0,\infty) \to [0,\infty), \qquad (\Phi')^{-1}(t) = \sup\{s : \Phi'(s) \leq t\}.$$

If $\Phi'$ is continuous and strictly increasing in $[0,\infty)$, then $(\Phi')^{-1}$ is the inverse function of $\Phi'$ and vice-versa [39, Remark 4.3.3]. We now define [39, Definition 4.3.1]

$$\Phi^* : [0,\infty) \to [0,\infty), \qquad \Phi^*(t) = \int_0^t (\Phi')^{-1}(s)\mathrm{d}s.$$

The function $\Phi^*$ is called the *complementary function* of $\Phi$. We note that $\Phi^*$ is also an $N$-function, $(\Phi^*)'(t) = (\Phi')^{-1}(t)$ for $t > 0$, and $(\Phi^*)^* = \Phi$.

Example 1 (the $p$-Laplacian). The following examples are important in our context. We let $p \in (1,\infty)$ and $\kappa \geq 0$. We define the $N$-functions $\Phi_p : [0,\infty) \to [0,\infty)$ and $\Phi_{p,\kappa} : [0,\infty) \to [0,\infty)$ as

$$\Phi_p(t) = \frac{1}{p}t^p, \qquad \Phi_{p,\kappa}(t) = \int_0^t \Phi'_{p,\kappa}(s)\mathrm{d}s, \qquad \Phi'_{p,\kappa}(t) = (\kappa + t)^{p-2}t. \tag{2.1}$$

We note that both $\Phi_{p,\kappa}$ and $\Phi_p$ satisfy the $\Delta_2$-condition with $\Delta_2(\Phi_{p,\kappa}) \leq C2^{\max\{2,p\}}$ and $\Delta_2(\Phi_p) = 2^p$; see [5, page 376]. Note that the first estimate is independent of $\kappa$. The corresponding complementary functions are

$$\Phi_p^*(t) = \frac{1}{p'}t^{p'}, \qquad \Phi^*_{p,\kappa}(t) \simeq (\kappa^{p-1} + t)^{p'-2}t^2, \qquad t \in [0,\infty), \qquad \frac{1}{p} + \frac{1}{p'} = 1.$$

According to [5, page 376], we have that $\Delta_2(\Phi_p^*) = 2^{p'}$ and $\Delta_2(\Phi^*_{p,\kappa}) \leq C2^{\max\{2,p'\}}$.

**2.1.2. Young's inequalities.** If $(\Phi, \Phi^*)$ is a pair of complementary $N$-functions, then we have

$$st \leq \Phi(s) + \Phi^*(t) \tag{2.2}$$

for all $s, t \geq 0$. The equality holds if and only if $t = \Phi'(s)$ or $s = (\Phi')^{-1}(t)$ [39, Theorem 4.3.4]. This is commonly referred to as *Young's inequality*. We also have the following refined versions: If $\Phi, \Phi^* \in \Delta_2$, then for all $\delta > 0$ there exists $C_\delta > 0$, depending on $\delta$, $\Delta_2(\Phi)$, and $\Delta_2(\Phi^*)$, such that [15, Lemma 32]

$$st \leq \delta\Phi(s) + C_\delta\Phi^*(t) \quad \forall s, t \geq 0, \tag{2.3}$$

$$s\Phi'(t) + \Phi'(s)t \leq \delta\Phi(s) + C_\delta\Phi(t) \quad \forall s, t \geq 0. \tag{2.4}$$

Note that the inequality (2.2) is a generalization of the classical Young's inequality: If $p \in (1,\infty)$ and $a, b \geq 0$, then

$$ab \leq \frac{1}{p}a^p + \frac{1}{p'}b^{p'}, \qquad \frac{1}{p} + \frac{1}{p'} = 1.$$

Similarly, (2.3) is a generalization of "Young's inequality with $\delta$": If $p \in (1,\infty)$ and $a, b \geq 0$, then for all $\delta > 0$ there exists $C_\delta > 0$ such that

$$ab \leq \delta a^p + C_\delta b^{p'}.$$

**2.1.3. Shifted $N$-functions.** Given an $N$-function $\Phi$ with $\Phi, \Phi^* \in \Delta_2$, we introduce the family of *shifted* functions $\{\Phi_a\}_{a\geq 0}$ as in [15, Definition 22]:

$$\Phi_a : [0,\infty) \to [0,\infty), \quad \Phi_a(t) = \int_0^t \Phi'_a(s)\mathrm{d}s, \quad \Phi'_a(t) := \Phi'(a+t)\frac{t}{a+t}, \ t \geq 0. \tag{2.5}$$

For all $a \geq 0$, $\Phi_a$ and $\Phi_a^*$ are $N$-functions and $\Phi_a, \Phi_a^* \in \Delta_2$. More importantly,

$$\sup_{a\geq 0} \{\Delta_2(\Phi_a)\}_{a\geq 0} \cup \{\Delta_2(\Phi_a^*)\}_{a\geq 0} \leq C(\Delta_2(\Phi), \Delta_2(\Phi^*));$$

see [15, Lemmas 23 and 27] and [17, Lemma 6.1].

Example 2 (the shifted $p$-Laplacian). Recall the function $\Phi_{p,\kappa}$ introduced in (2.1). In this case we have, for $a \geq 0$ and $t \geq 0$,

$$\Phi_{p,\kappa,a}(t) \simeq (\kappa + a + t)^{p-2} t^2, \quad \Phi_{p,\kappa,a}^*(t) \simeq ((\kappa + a)^{p-1} + t)^{p'-2} t^2.$$

For all $a \geq 0$, $\Delta_2(\Phi_{p,\kappa,a}) \leq C2^{\max\{2,p\}}$ and $\Delta_2(\Phi_{p,\kappa,a}^*) \leq C2^{\max\{2,p\}}$. Thus, $\{\Phi_a\}_{a\geq 0}$ and $\{\Phi_a^*\}_{a\geq 0}$ satisfy the $\Delta_2$-condition uniformly with respect to $a$; see [5, page 376].

**2.1.4. Structural assumptions.** In order to be able to analyze our problems, we assume that the function $\varphi$ occurring in (1.1) is indeed an $N$-function. Moreover, we have to assume that $\varphi$ satisfies the following condition; see [5, assumption A.1] and [6, assumption 2.1].

Assumption 2.1 (equivalence). Let $\Phi$ be an $N$-function. We assume that $\Phi$ is $C^1$ in $[0, \infty)$, $\Phi$ is $C^2$ on $(0, \infty)$, and

$$\Phi'(t) \simeq t\,\Phi''(t) \tag{2.6}$$

uniformly in $t > 0$. The constants hidden in $\simeq$ are called the characteristics of $\Phi$.

The following consequences of Assumption 2.1 are important. First, $\Phi$ is strictly convex in $(0, \infty)$, $\Phi'$ is strictly monotone increasing in $(0, \infty)$, and $\Phi$ satisfies the $\Delta_2$-condition [6, page 487]. Moreover, $\Delta_2(\Phi)$ depends only on the characteristics of $\Phi$. Second, $\Phi^*$ also satisfies Assumption 2.1 [15, Lemma 25]. In particular, $\Delta_2(\Phi^*) < \infty$. Third, uniformly in $s, t \in \mathbb{R}$, we have the following properties [15, Lemma 24]:

$$\Phi''(|s| + |t|)|s - t| \simeq \Phi'_{|s|}(|s - t|), \qquad \Phi''(|s| + |t|)|s - t|^2 \simeq \Phi_{|s|}(|s - t|). \tag{2.7}$$

Fourth, the following shifted version of Young's inequality holds: for all $\delta > 0$ there exists a positive constant $C_\delta$ such that [15, Lemma 32]

$$s\Phi'_a(t) + \Phi'_a(s)t \leq \delta\Phi_a(s) + C_\delta\Phi_a(t), \qquad \forall s, t, a \geq 0. \tag{2.8}$$

**2.2. The operator A.** Let $\Phi$ be an $N$-function that satisfies Assumption 2.1. We define the nonlinear vector field $\mathbf{A}$ as follows:

$$\mathbf{A} : \mathbb{R}^d \to \mathbb{R}^d, \qquad \mathbf{A}(\boldsymbol{\zeta}) := \Phi'(|\boldsymbol{\zeta}|)\frac{\boldsymbol{\zeta}}{|\boldsymbol{\zeta}|} \quad \forall \boldsymbol{\zeta} \in \mathbb{R}^d. \tag{2.9}$$

Example 3 (the $p$-Laplacian). We let $p \in (1, \infty)$ and $\kappa \geq 0$. Recall the function $\Phi_{p,\kappa}$ introduced in (2.1). Note that $\Phi_{p,\kappa}$ satisfies the conditions in Assumption 2.1 and that the characteristics of $\Phi_{p,\kappa}$ do not depend on $\kappa$. In fact, we have

$$\min\{1, p - 1\}(\kappa + t)^{p-2} \leq \Phi''_{p,\kappa}(t) \leq \max\{1, p - 1\}(\kappa + t)^{p-2}$$

for every $t > 0$ [5, page 376]. In this scenario

$$\mathbf{A}(\boldsymbol{\zeta}) = (\kappa + |\boldsymbol{\zeta}|)^{p-2}\boldsymbol{\zeta}, \quad \boldsymbol{\zeta} \in \mathbb{R}^d.$$

*Remark* 2.2 ($\Phi$-monotonicity and $\Phi$-growth). It is shown in [15, Lemma 21] that there exist positive constants $C_0$ and $C_1$ such that

$$(\mathbf{A}(\boldsymbol{\zeta}) - \mathbf{A}(\boldsymbol{\eta})) \cdot (\boldsymbol{\zeta} - \boldsymbol{\eta}) \geq C_0 \Phi''(|\boldsymbol{\zeta}| + |\boldsymbol{\eta}|)|\boldsymbol{\zeta} - \boldsymbol{\eta}|^2 \quad \forall \boldsymbol{\zeta}, \boldsymbol{\eta} \in \mathbb{R}^d, \tag{2.10}$$

$$|\mathbf{A}(\boldsymbol{\zeta}) - \mathbf{A}(\boldsymbol{\eta})| \leq C_1 \Phi''(|\boldsymbol{\zeta}| + |\boldsymbol{\eta}|)|\boldsymbol{\zeta} - \boldsymbol{\eta}| \quad \forall \boldsymbol{\zeta}, \boldsymbol{\eta} \in \mathbb{R}^d. \tag{2.11}$$

Here, $C_0$ and $C_1$ depend only on $\Delta_2(\Phi)$, $\Delta_2(\Phi^*)$, and the characteristics of $\Phi$.

Let us now define the $N$-function $\Psi : [0, \infty) \to [0, \infty)$ by

$$\Psi(0) = 0, \qquad \Psi'(t) \coloneqq \sqrt{t\Phi'(t)}, \quad t > 0.$$

It is shown in [15, Lemma 25] that the $N$-functions $\Psi$ and $\Psi^*$ also satisfy Assumption 2.1 and that $\Psi''(t) \simeq \sqrt{\Phi''(t)}$ uniformly in $t > 0$. With the function $\Psi$ at hand, we introduce the vector field $\mathbf{V}$ as follows:

$$\mathbf{V} : \mathbb{R}^d \to \mathbb{R}^d, \qquad \mathbf{V}(\boldsymbol{\zeta}) \coloneqq \Psi'(|\boldsymbol{\zeta}|)\frac{\boldsymbol{\zeta}}{|\boldsymbol{\zeta}|} \quad \forall \boldsymbol{\zeta} \in \mathbb{R}^d. \tag{2.12}$$

An application of [15, Lemma 21] shows that the bounds (2.10) and (2.11) hold if $\mathbf{A}$ and $\Phi$ are replaced by $\mathbf{V}$ and $\Psi$, respectively.

Example 4 (the $p$-Laplacian). We let $p \in (1, \infty)$ and $\kappa \geq 0$. The following functions correspond to those defined in (2.1):

$$\Psi_p'(t) = t^{\frac{p}{2}}, \quad \mathbf{V}_p(\boldsymbol{\zeta}) = |\boldsymbol{\zeta}|^{\frac{p-2}{2}}\boldsymbol{\zeta}, \quad \Psi_{p,\kappa}'(t) = (\kappa + t)^{\frac{p-2}{2}}t, \quad \mathbf{V}_{p,\kappa}(\boldsymbol{\zeta}) = (\kappa + |\boldsymbol{\zeta}|)^{\frac{p-2}{2}}\boldsymbol{\zeta}.$$

The following result expresses the relationship between $\mathbf{A}$, $\mathbf{V}$, and $\{\Phi_a\}_{a\geq 0}$.

Proposition 2.3 (equivalences). *Let* $\mathbf{A}$ *and* $\mathbf{V}$ *be defined as in* (2.9) *and* (2.12), *respectively. Then, for all* $\boldsymbol{\zeta}, \boldsymbol{\eta} \in \mathbb{R}^d$, *we have*

$$(\mathbf{A}(\boldsymbol{\zeta}) - \mathbf{A}(\boldsymbol{\eta})) \cdot (\boldsymbol{\zeta} - \boldsymbol{\eta}) \simeq |\mathbf{V}(\boldsymbol{\zeta}) - \mathbf{V}(\boldsymbol{\eta})|^2 \tag{2.13}$$

$$\simeq \Phi_{|\boldsymbol{\zeta}|}(|\boldsymbol{\zeta} - \boldsymbol{\eta}|) \tag{2.14}$$

$$\simeq \Phi_{|\boldsymbol{\eta}|}(|\boldsymbol{\zeta} - \boldsymbol{\eta}|) \tag{2.15}$$

$$\simeq |\boldsymbol{\zeta} - \boldsymbol{\eta}|^2 \Phi''(|\boldsymbol{\zeta}| + |\boldsymbol{\eta}|). \tag{2.16}$$

*Moreover, for all* $\boldsymbol{\zeta}, \boldsymbol{\eta} \in \mathbb{R}^d$, *we have*

$$|\mathbf{A}(\boldsymbol{\zeta}) - \mathbf{A}(\boldsymbol{\eta})| \simeq \Phi_{|\boldsymbol{\zeta}|}'(|\boldsymbol{\zeta} - \boldsymbol{\eta}|). \tag{2.17}$$

*The constants hidden in* $\simeq$ *only depend on the characteristics of* $\Phi$.

*Proof.* A proof of the estimates (2.13)–(2.16) can be found in [15, Lemma 3] while a proof for (2.17) can be found in [34, Corollary 64]. ∎

**2.3. Orlicz spaces.** Let $\mu$ be a Borel measure on $\mathbb{R}^d$ that is absolutely continuous with respect to the Lebesgue measure. We denote by $L^0(\Omega)$ the set of all Lebesgue measurable functions in $\Omega$. Let $\Phi$ be an $N$-function. We define $\varrho : L^0(\Omega) \to \mathbb{R}$ by

$$\varrho(f, \Phi) \coloneqq \int_\Omega \Phi(|f(x)|)\mathrm{d}\mu(x).$$

The function $\varrho$ is a semimodular on $L^0(\Omega)$. Moreover, if $\Phi$ is positive, then $\varrho$ is a modular [16, Lemma 2.3.10]. We define the *Orlicz space* [40, Sec. 3.1, Definition 5]

$$L^\Phi(\mu, \Omega) \coloneqq \left\{ v \in L^0(\Omega) : \exists k > 0 : \varrho(kv, \Phi) < \infty \right\}$$

equipped with the Luxemburg norm (see [40, Sec. 3.2, (6) and Theorem 3])

$$\|f\|_{L^\Phi(\mu,\Omega)} \coloneqq \inf\left\{k>0 : \varrho\left(\frac{v}{k},\Phi\right)\leq 1\right\}. \tag{2.18}$$

The following is a list of properties of the space $L^\Phi(\mu,\Omega)$:

- $L^\Phi(\mu,\Omega)$ is a Banach space [40, Sec. 3.3, Theorem 10].
- If $\Phi\in\Delta_2$, then (see [40, Sec. 3.1, Theorem 2 and Sec. 3.3, Proposition 3])
$$L^\Phi(\mu,\Omega)=\left\{v\in L^0(\Omega):\varrho(v,\Phi)<\infty\right\}.$$
- If $\Phi\in\Delta_2$, then $L^\Phi(\mu,\Omega)$ is separable [40, Sec. 3.5, Thm. 1 and Sec. 3.4, Cor. 5].
- If $(\Phi,\Phi^*)$ is a pair of complementary $N$-functions, then we have Hölder's inequality [40, Sec. 3.3, Proposition 1] (the constant 2 cannot be omitted):
$$\int_\Omega |f(x)g(x)|\mathrm{d}\mu(x)\leq 2\|f\|_{L^\Phi(\mu,\Omega)}\|g\|_{L^{\Phi^*}(\mu,\Omega)} \qquad \forall f\in L^\Phi(\mu,\Omega),\ \forall g\in L^{\Phi^*}(\mu,\Omega).$$
- If $(\Phi,\Phi^*)$ is a pair of complementary $N$-functions with $\Phi,\Phi^*\in\Delta_2$, then the dual space $[L^\Phi(\mu,\Omega)]^*$ of $L^\Phi(\mu,\Omega)$ is isomorphic to the space $L^{\Phi^*}(\mu,\Omega)$ in the following sense [40, Sec. 4.1, Theorem 7]:
$$w\in L^{\Phi^*}(\mu,\Omega)\mapsto J_w\in[L^\Phi(\mu,\Omega)]^*: \quad J_w(v)\coloneqq\int_\Omega v(x)w(x)\mathrm{d}\mu(x),\ v\in L^\Phi(\mu,\Omega),$$
and $\|J_w\|_{L^\Phi(\mu,\Omega)^*}\leq 2\|w\|_{L^{\Phi^*}(\mu,\Omega)}$. $L^\Phi(\mu,\Omega)$ is reflexive [40, Sec. 4.1, Thm. 10].

We refer the reader to [33, 40, 16, 30, 36] for further facts and properties of Orlicz spaces and their generalizations.

**2.4. $A_p$ weights.** A weight $\omega:\mathbb{R}^d\to\mathbb{R}$ is a locally integrable function in $\mathbb{R}^d$ which is such that $\omega(x)>0$ for a.e. $x\in\mathbb{R}^d$. For a weight $\omega$ and a measurable set $A\subset\mathbb{R}^d$, we define $\omega(A)\coloneqq\int_A\omega(x)\,\mathrm{d}x$. For $p\in[1,\infty)$ we say that a weight $\omega$ belongs to the Muckenhoupt class $A_p$ if there is a positive constant $C$ such that for every ball $B\subset\mathbb{R}^d$ [43, Definition 1.2.2]

$$\left(\fint_B\omega(x)\,\mathrm{d}x\right)\left(\fint_B\omega(x)^{-\frac{1}{p-1}}\,\mathrm{d}x\right)^{p-1}\leq C, \qquad p>1, \tag{2.19}$$

$$\left(\fint_B\omega(x)\,\mathrm{d}x\right)\sup_{x\in B}\frac{1}{\omega(x)}\leq C, \qquad p=1. \tag{2.20}$$

The infimum over all such constants $C$ is called the Muckenhoupt characteristic of $\omega$ and is denoted by $[\omega]_{A_p}$. We define the class $A_\infty$ as $A_\infty=\cup_{p\geq1}A_p$. We note that

$$\omega\in A_p \iff \omega'\coloneqq\omega^{-\frac{1}{p-1}}\in A_{p'}, \qquad [\omega]_{A_p}=[\omega']_{A_{p'}}^{p-1}, \qquad p\in(1,\infty);$$

see [43, Remark 1.2.4, item 4], where we have denoted the Hölder conjugate of $p$ by $p'$. We refer the reader to [43, 19, 32, 28] for more details on Muckenhoupt weights.

Many useful properties follow from the fact that $\omega\in A_p$. We mention here one that will be useful for us in the following:

- *Open ended property*: If $\omega\in A_p$ with $p\in(1,\infty)$, then there is $\delta>0$ such that $\omega\in A_{p-\delta}$; see [43, Corollary 1.2.17] and [19, Corollary 7.6, item (2)].

We refer the reader to [43, 19, 32, 28, 14] for more details on Muckenhoupt weights.

In dealing with the obstacle problem, we restrict ourselves to a class of weights that behave well near the boundary of the domain. The following definition is inspired by [24, Definition 2.5].

DEFINITION 2.4 (class $A_p(\Omega)$). *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain and let $p \in [1, \infty)$. We say that $\omega \in A_p$ belongs to the class $A_p(\Omega)$ if there is an open set $\mathcal{G} \subset \Omega$, and $\varepsilon, \omega_l > 0$ such that*

$$\{x \in \Omega : \operatorname{dist}(x, \partial\Omega) < \varepsilon\} \subset \mathcal{G}, \qquad \omega_{|\bar{\mathcal{G}}} \in C(\bar{\mathcal{G}}), \qquad \inf_{x \in \bar{\mathcal{G}}} \omega(x) \geq \omega_l.$$

**2.5. Weighted Orlicz spaces.** We have finally reached the point where we can describe the functional framework that we will use in our analysis. Namely, that of *weighted Orlicz spaces*. We let $\omega \in A_1$ and define the measure $\mu$ via $\mathrm{d}\mu(x) = \omega(x)\,\mathrm{d}x$. Given an $N$-function $\Phi$, we define the weighted Orlicz space

$$L^\Phi(\omega, \Omega) := \left\{ v \in L^0(\Omega) : \exists k > 0 : \ \varrho(kv, \Phi) := \int_\Omega \omega(x)\Phi(k|v(x)|)\mathrm{d}x < \infty \right\}.$$

We endow $L^\Phi(\omega, \Omega)$ with the Luxemburg norm defined in (2.18), and we refer to Section 2.3 for a list of properties of this space.

Let us now assume that $L^\Phi(\omega, \Omega) \subset L^1_{\mathrm{loc}}(\Omega)$. Then, the elements of $L^\Phi(\omega, \Omega)$ are distributions and they have distributional derivatives. We can then define the *weighted Orlicz–Sobolev space* [36, Definition 3.1.1]

$$W^{1,\Phi}(\omega, \Omega) := \{v \in L^\Phi(\omega, \Omega) : \partial_i v \in L^\Phi(\omega, \Omega) \ \forall i \in \{1, \ldots, d\}\}$$

endowed with the norm $\|v\|_{W^{1,\Phi}(\omega,\Omega)} := \|v\|_{L^\Phi(\omega,\Omega)} + \|\nabla v\|_{L^\Phi(\omega,\Omega)}$. Many of the properties of the classical Sobolev spaces extend to the weighted Orlicz–Sobolev setting:
1. $W^{1,\Phi}(\omega, \Omega)$ is a Banach space [30, Theorem 6.1.4(b)].
2. If $\Phi \in \Delta_2$, then $W^{1,\Phi}(\omega, \Omega)$ is separable [30, Theorem 6.1.4(c)].
3. If $\Phi, \Phi^* \in \Delta_2$, then $W^{1,\Phi}(\omega, \Omega)$ is reflexive [30, Theorem 6.1.4(d)].

We define $W^{1,\Phi}_0(\omega, \Omega)$ as the closure of $C_0^\infty(\Omega) \cap W^{1,\Phi}(\omega, \Omega)$ in $W^{1,\Phi}(\omega, \Omega)$ [30, Definition 6.1.8]. It follows that this space has the same properties as $W^{1,\Phi}(\omega, \Omega)$; see [30, Theorem 6.1.9]. In addition, the following modular Poincaré inequality holds.

PROPOSITION 2.5 (modular Poincaré inequality). *Let $\Phi$ be an $N$-function such that $\Phi, \Phi^* \in \Delta_2$. If $\omega \in A_1$, then*

$$\int_\Omega \omega(x)\Phi(|v(x)|)\mathrm{d}x \lesssim \int_\Omega \omega(x)\Phi(|\nabla v(x)|)\mathrm{d}x \quad \forall v \in W^{1,\Phi}_0(\omega, \Omega). \tag{2.21}$$

*Proof.* The result follows directly from [14, Theorem 4.15]. For the sake of completeness, we provide some context and explanations.

- [14, Theorem 4.15] begins by letting $\mathcal{B}$ be a Muckenhoupt basis that is $A_{p,\mathcal{B}}$ open. In our context, $\mathcal{B}$ is nothing but the set of all balls $B \subset \mathbb{R}^d$, so that $A_{p,\mathcal{B}} = A_p$, the standard Muckenhoupt class. Thus, the fact that $A_{p,\mathcal{B}}$ is open is the open ended property of the class $A_p$ stated at the end of Section 2.4.
- The next step is to provide a class $\mathcal{F}$ of pairs $(f, g)$ of nonnegative measurable functions that are not identically zero, so that for some $p_0 \in [1, \infty)$ and all $w_0 \in A_{p_0}$

$$\int_{\mathbb{R}^d} f(x)^{p_0} w_0(x)\,\mathrm{d}x \leq C \int_{\mathbb{R}^d} g(x)^{p_0} w_0(x)\,\mathrm{d}x.$$

If we set $\mathcal{F} := \{(|w|, |\nabla w|) : w \in C_0^\infty(\Omega)\}$, the corresponding inequality is nothing but a weighted Poincaré inequality that holds for every $p_0 \in (1,\infty)$ and all $w_0 \in A_{p_0}$; see [23, Theorem 1.3].

- As we do here, [14, Theorem 4.15] assumes that $\Phi$ is an $N$-function with $\Phi, \Phi^* \in \Delta_2$. We note that in [14] the complementary function of $\Phi$ is denoted by $\bar{\Phi}$.
- In [14, Theorem 4.15] it is assumed that the weight $\omega$ belongs to $A_{i(\Phi)}$. In our case, we assume that the weight belongs to $A_1$ and note that $A_1 \subset A_{i(\Phi)}$.

The conclusion is thus that (2.21) holds for all $v \in C_0^\infty(\Omega)$. Finally, we argue by density. This concludes the proof. ∎

*Remark* 2.6 (equivalence). On $W_0^{1,\Phi}(\omega,\Omega)$, $\|\nabla w\|_{L^\Phi(\omega,\Omega)}$ defines a norm that is equivalent to $\|w\|_{W^{1,\Phi}(\omega,\Omega)}$, provided that $\omega \in A_1$ and that $\Phi, \Phi^* \in \Delta_2$.

To conclude our discussion, we present a suboptimal sufficient condition for the inclusion $L^\Phi(\omega,\Omega) \subset L^1_{\text{loc}}(\Omega)$.

PROPOSITION 2.7 ($L^\Phi(\omega,\Omega) \subset L^1_{\text{loc}}(\Omega)$). *Let $\Phi$ be an $N$-function. If $\omega \in A_1$, then every function in $L^\Phi(\omega,\Omega)$ is locally integrable in $\Omega$.*

*Proof.* Let $K \subset \Omega$ be compact and let $v \in L^\Phi(\omega,\Omega)$. A direct application of the $A_1$-condition (2.20) shows that

$$\int_K |v(x)| \mathrm{d}x = \int_K \omega(x)^{-1}\omega(x)|v(x)| \mathrm{d}x \le \sup_{x\in K}\frac{1}{\omega(x)} \int_K \omega(x)|v(x)|\mathrm{d}x \lesssim \left( \fint_K \omega(x)\,\mathrm{d}x \right)^{-1} \int_K \omega(x)|v(x)|\mathrm{d}x.$$

Finally, we control $\int_K \omega(x)|v(x)|\mathrm{d}x$ using Young's inequality (2.2):

$$\int_K \omega(x)|v(x)|\mathrm{d}x \le \int_K \omega(x)\left[\Phi(|v(x)|) + \Phi^*(1)\right]\mathrm{d}x \lesssim \int_\Omega \omega(x)\Phi(|v(x)|)\mathrm{d}x + \Phi^*(1)\int_K \omega(x)\mathrm{d}x < \infty,$$

where we used in the last inequality that a weight is locally integrable in $\mathbb{R}^d$. ∎

**2.5.1. Weighted Lebesgue and Sobolev spaces.** A specific example of the constructions described above are weighted Lebesgue and Sobolev spaces. To introduce them, let $p \in (1,\infty)$ and $\omega \in A_p$. The weighted Lebesgue space $L^p(\omega,\Omega)$ is obtained by using the $N$-function $\Phi(t) = t^p/p$. We immediately note that $L^p(\omega,\Omega) \subset L^1_{\text{loc}}(\Omega)$ (see [43, bounds (1.2.1)–(1.2.2)]), so we define the weighted Sobolev spaces

$$W^{1,p}(\omega,\Omega) := \{w \in L^p(\omega,\Omega) : \partial_i w \in L^p(\omega,\Omega)\, \forall i \in \{1,\dots,d\}\}.$$

We define $W_0^{1,p}(\omega,\Omega)$ as the closure of $C_0^\infty(\Omega)$ in $W^{1,p}(\omega,\Omega)$. Due to the fact that $\omega \in A_p$ many of the properties of the classical Sobolev spaces extend to the weighted spaces [43, 26, 35]. In particular, we have a weighted Poincaré inequality: if $D \subset \mathbb{R}^d$ is open, bounded, and Lipschitz; $p \in (1,\infty)$; and $\omega \in A_p$, then [23, Theorem 1.3]

$$\|w\|_{L^p(\omega,D)} \le C_p \operatorname{diam}(D)\|\nabla w\|_{L^p(\omega,D)} \quad \forall w \in W_0^{1,p}(\omega,D). \tag{2.22}$$

The constant $C_p$ depends on $\omega$ only through $[\omega]_{A_p}$. The following weighted Poincaré inequality is also useful for our analysis [37, Lemmas 3.1 and 4.2]: There exists a constant $w_D$ such that

$$\|w - w_D\|_{L^p(\omega,D)} \le C_p \operatorname{diam}(D)\|\nabla w\|_{L^p(\omega,D)} \quad \forall w \in W^{1,p}(\omega,D). \tag{2.23}$$

Finally, we recall a scaled trace inequality. Let $T \subset \mathbb{R}^d$ be a simplex and $F \subset T$ a face of $T$. If $p \in [1, \infty)$ and $w \in W^{1,p}(T)$, then its trace $\operatorname{tr}_{\partial T} w \in L^p(\partial T)$ satisfies

$$\frac{1}{|F|}\| \operatorname{tr}_F w\|^p_{L^p(F)} \lesssim \frac{1}{|T|}\|w\|^p_{L^p(T)} + \frac{\operatorname{diam}(T)^p}{|T|}\|\nabla w\|^p_{L^p(T)}. \tag{2.24}$$

**3. The weighted $\varphi$-Laplace problem.** We now present the assumptions that allow us to properly formulate and analyze the problem of minimizing (1.1). Throughout our discussion, $\Omega \subset \mathbb{R}^d$ is a Lipschitz polytope. We let $\omega \in A_1$ and let $\varphi$ be an $N$-function that satisfies the conditions in Assumption 2.1.

For $f \in L^{\varphi^*}(\omega, \Omega)$, the following variational problem is a necessary and sufficient condition for a minimum of $\mathcal{J}$ over $W^{1,\varphi}_0(\omega, \Omega)$: Find $u \in W^{1,\varphi}_0(\omega, \Omega)$ such that

$$\int_\Omega \omega(x)\mathbf{A}(\nabla u(x)) \cdot \nabla v(x)\mathrm{d}x = \int_\Omega \omega(x)f(x)v(x)\mathrm{d}x \quad \forall v \in W^{1,\varphi}_0(\omega, \Omega), \tag{3.1}$$

where $\mathbf{A}$ is defined in (2.9). We note that, on the basis of Youngs inequality, we have

$$\left|\int_\Omega \omega(x)f(x)v(x)\mathrm{d}x\right| \leq \int_\Omega \omega(x)\varphi(|v(x)|)\mathrm{d}x + \int_\Omega \omega(x)\varphi^*(|f(x)|)\mathrm{d}x < \infty,$$

so that the term on the right hand side of (3.1) is well-defined. A convex minimization argument or the theory of monotone operators guarantee the existence and uniqueness of a weak solution.

**3.1. Finite element discretization.** Now that we have established the well-posedness of (3.1), we proceed with its approximation. Since we have assumed that $\Omega$ is a polytope, it can be meshed exactly. We denote by $\mathscr{T}_h = \{T\}$ a conforming partition, or mesh, of $\bar{\Omega}$ into closed simplices $T$ of size $h_T = \operatorname{diam}(T)$. Here, $h = \max\{h_T : T \in \mathscr{T}_h\}$. For $T \in \mathscr{T}_h$, we define $S_T$ as the set of all elements in $\mathscr{T}_h$ that share at least one vertex with $T$. We denote by $\mathbb{T} = \{\mathscr{T}_h\}_{h>0}$ a collection of conforming meshes that are refinements of an initial mesh $\mathscr{T}_0$. We assume that $\mathbb{T}$ satisfies the so-called shape regularity condition [22, Definition 1.107]: there exits a constant $\sigma > 1$ such that

$$\sup_{\mathscr{T}_h \in \mathbb{T}} \max\{\sigma_T : T \in \mathscr{T}_h\} \leq \sigma,$$

where $\sigma_T := h_T/\rho_T$ is the shape coefficient of $T$ and $\rho_T$ is the diameter of the sphere inscribed in $T$. With this setting, given $\mathscr{T}_h \in \mathbb{T}$, we define the finite element spaces

$$\mathbb{W}_h := \{v_h \in W^{1,\infty}(\Omega) : \ v_{h|T} \in \mathbb{P}_1(T) \ \forall T \in \mathscr{T}_h\}, \qquad \mathbb{V}_h := \mathbb{W}_h \cap W^{1,\infty}_0(\Omega).$$

The finite element approximation of (3.1) is: Find $u_h \in \mathbb{V}_h$ such that

$$\int_\Omega \omega(x)\mathbf{A}(\nabla u_h(x)) \cdot \nabla v_h(x)\mathrm{d}x = \int_\Omega \omega(x)f(x)v_h(x)\mathrm{d}x \quad \forall v_h \in \mathbb{V}_h. \tag{3.2}$$

Once again, a convex minimization argument immediately yields the existence and uniqueness of a finite element solution.

**3.2. A best approximation result.** We present a best approximation result for the finite element scheme (3.2). We begin by noting that a direct property of the finite element approximation is Galerkin orthogonality:

$$\int_\Omega \omega(x)\left(\mathbf{A}(\nabla u(x)) - \mathbf{A}(\nabla u_h(x))\right) \cdot \nabla v_h(x)\mathrm{d}x = 0 \quad \forall v_h \in \mathbb{V}_h. \tag{3.3}$$

This is the first step towards proving *best approximation* for finite element solutions.

THEOREM 3.1 (best approximation). *Let $u \in W_0^{1,\varphi}(\omega,\Omega)$ solve (3.1), and let $u_h \in \mathbb{V}_h$ be its finite element approximation defined in (3.2). Then,*

$$\|\mathbf{V}(\nabla u) - \mathbf{V}(\nabla u_h)\|_{L^2(\omega,\Omega)} \lesssim \inf_{v_h \in \mathbb{V}_h} \|\mathbf{V}(\nabla u) - \mathbf{V}(\nabla v_h)\|_{L^2(\omega,\Omega)}. \tag{3.4}$$

*Proof.* The derivation of the best approximation property (3.4) modifies the arguments in [17, Lemma 5.2] to account for our weighted setting. To proceed, we define $e = u - u_h$. Moreover, for an arbitrary but fixed $v_h \in \mathbb{V}_h$, we set $z = u - v_h$. We begin by using the equivalence (2.13) and Galerkin orthogonality to obtain

$$\begin{aligned}\mathfrak{I} &:= \int_\Omega \omega(x)|\mathbf{V}(\nabla u(x)) - \mathbf{V}(\nabla u_h(x))|^2 \mathrm{d}x \\ &\simeq \int_\Omega \omega(x)(\mathbf{A}(\nabla u(x)) - \mathbf{A}(\nabla u_h(x))) \cdot \nabla e(x) \mathrm{d}x \\ &= \int_\Omega \omega(x)(\mathbf{A}(\nabla u(x)) - \mathbf{A}(\nabla u_h(x))) \cdot \nabla z(x)) \mathrm{d}x.\end{aligned}$$

We now use the equivalences (2.16) and (2.7) to derive

$$\begin{aligned}\mathfrak{I} &\simeq \int_\Omega \omega(x)\varphi''(|\nabla u(x)| + |\nabla u_h(x)|)|\nabla e(x)||\nabla z(x)|\mathrm{d}x \\ &\simeq \int_\Omega \omega(x)\varphi'_{|\nabla u(x)|}(|\nabla e(x)|)|\nabla z(x)|\mathrm{d}x.\end{aligned}$$

An application of the shifted version of Young's inequality (2.8) shows that for all $\delta > 0$ there exists a positive constant $C_\delta$ so that

$$\begin{aligned}\mathfrak{I} &\leq C\delta \int_\Omega \omega(x)\varphi_{|\nabla u(x)|}(|\nabla e(x)|)\mathrm{d}x + C_\delta \int_\Omega \omega(x)\varphi_{|\nabla u(x)|}(|\nabla z(x)|)\mathrm{d}x \\ &\leq C\delta \int_\Omega \omega(x)|\mathbf{V}(\nabla u(x)) - \mathbf{V}(\nabla u_h(x))|^2 \mathrm{d}x + C_\delta \int_\Omega \omega(x)\varphi_{|\nabla u(x)|}(|\nabla z(x)|)\mathrm{d}x,\end{aligned}$$

where to obtain the last inequality we used the equivalence (2.14). Using the definition of $\mathfrak{I}$ and the equivalence (2.14) once again we obtain

$$\mathfrak{I} \leq C\delta\mathfrak{I} + CC_\delta\|\mathbf{V}(\nabla u) - \mathbf{V}(\nabla v_h)\|^2_{L^2(\omega,\Omega)}.$$

The term $C\delta\mathfrak{I}$ appearing in the previous inequality can be absorbed on the left hand side if $\delta$ is chosen carefully. This gives us the estimate (3.4) and concludes the proof. □

**3.3. Stability estimates for an interpolation operator.** In this section, we derive stability estimates in weighted Orlicz–Sobolev spaces for a suitable interpolation operator $\Pi_h$. As in [17], we assume that such an interpolation operator

$$\Pi_h : W^{1,1}(\Omega) \to \mathbb{W}_h$$

satisfies the following two properties:

1. *Projection:* $\Pi_h$ is a projection, i.e., $\Pi_h w_h = w_h$ for all $w_h \in \mathbb{W}_h$.
2. *Stability:* Let $v \in W^{1,1}(\Omega)$. Then, for each $T \in \mathscr{T}_h$ we have the bound

$$\|\nabla \Pi_h v\|_{L^1(T)} \lesssim \|\nabla v\|_{L^1(S_T)}. \tag{3.5}$$

Recall that $S_T$ is the set of elements in $\mathscr{T}_h$ that share at least one vertex with $T$.

An example of an interpolation operator $\Pi_h$ that satisfies these two properties is the so-called Scott–Zhang operator, which was introduced in [42]. The construction of $\Pi_h$ is such that it preserves homogeneous boundary conditions, i.e., $v = 0$ on $\partial\Omega$ implies $\Pi_h v = 0$ on $\partial\Omega$ [42, (2.17)]. Moreover, $\Pi_h$ is a projection from $W^{1,1}(\Omega)$ to $\mathbb{W}_h$ with the property that $W_0^{1,1}(\Omega)$ is mapped to $\mathbb{V}_h$ [42, Theorem 2.1]. The Scott–Zhang operator also satisfies the stability bound (3.5). We refer the reader to [42, Theorem 3.1] for a proof of this result and mention that in this case the hidden constant in (3.5) depends only on the shape-regularity coefficient $\sigma$.

Inspired by the unweighted case treated in [17, Theorem 4.5], we present a stability result in weighted Sobolev–Orlicz spaces $W^{1,\Phi}(\omega,\Omega)$.

Theorem 3.2 (weighted stability). *Let $\Phi$ be an $N$-function satisfying the conditions in Assumption 2.1, let $\omega \in A_1$, and let $v \in W^{1,\Phi}(\omega,\Omega)$. Then, for each $T \in \mathscr{T}_h$ we have the bound*

$$\fint_T \omega(x)\Phi(|\nabla \Pi_h v(x)|)\mathrm{d}x \lesssim \fint_{S_T} \omega(x)\Phi(|\nabla v(x)|)\mathrm{d}x, \tag{3.6}$$

*where the hidden constant depends on $\sigma$, $\Delta_2(\Phi)$, and $[\omega]_{A_1}$.*

*Proof.* Let $T \in \mathscr{T}_h$ and let $x \in T$. We begin the proof by noting that

$$|\nabla \Pi_h v(x)| \lesssim \fint_T |\nabla \Pi_h v(z)|\mathrm{d}z \le C \fint_{S_T} |\nabla v(z)|\mathrm{d}z.$$

The first bound follows from the fact that $(\nabla \Pi_h v)_{|K} \in \mathbb{P}_0(T)$, while the second estimate follows from the stability of the interpolation operator $\Pi_h$ in (3.5). We note that $C$ depends on the shape-regularity coefficient $\sigma$. We can therefore rely on the monotonicity of $\Phi$ and the $\Delta_2$-condition, which holds because $\Phi$ satisfies the conditions in Assumption 2.1, to obtain

$$\begin{aligned}\fint_T \omega(x)\Phi(|\nabla \Pi_h v(x)|)\mathrm{d}x &\le \fint_T \omega(x)\Phi\left(C\fint_{S_T}|\nabla v(z)|\mathrm{d}z\right)\mathrm{d}x \\ &\lesssim \fint_T \omega(x)\Phi\left(\fint_{S_T}|\nabla v(z)|\mathrm{d}z\right)\mathrm{d}x.\end{aligned}$$

Next, we use Jensens inequality and the fact that the weight $\omega$ satisfies the $A_1$-condition to derive

$$\begin{aligned}\fint_T \omega(x)\Phi(|\nabla \Pi_h v(x)|)\mathrm{d}x &\lesssim \fint_T \omega(x)\fint_{S_T}\Phi(|\nabla v(z)|)\mathrm{d}z\mathrm{d}x \\ &\le C\fint_T \omega(x)\mathrm{d}x\left(\sup_{z\in S_T}\frac{1}{\omega(z)}\right)\fint_{S_T}\omega(z)\Phi(|\nabla v(z)|)\mathrm{d}z.\end{aligned}$$

Since $T \subset S_T$ with $|T| \simeq |S_T|$, an application of the $A_1$-condition (2.20) allows us to conclude the desired stability estimate. ∎

Next, we present the result of Theorem 3.2 in the context of a family of shifted $N$-functions.

Corollary 3.3 (weighted stability). *Let $\omega \in A_1$, and let $\{\Phi_a\}_{a\ge 0}$ be the family of shifted $N$-functions defined in (2.5) associated with the $N$-function $\Phi$ that satisfies the conditions in Assumption 2.1. Then, we have the following stability bound* *uniformly in $a \geq 0$ and $T \in \mathscr{T}_h$:*

$$\fint_T \omega(x)\Phi_a(|\nabla \Pi_h v(x)|)\mathrm{d}x \lesssim \fint_{S_T} \omega(x)\Phi_a(|\nabla v(x)|)\mathrm{d}x, \tag{3.7}$$

*where the hidden constant depends on $\sigma$, $\Delta_2(\Phi)$, and $[\omega]_{A_1}$.*

*Proof.* The desired bound results from the application of the stability bound of Theorem 3.2 to the family of shifted $N$-functions in conjunction with the fact that such a family satisfies the $\Delta_2$-condition uniformly in $a \geq 0$ with a constant that depends only on $\Delta_2(\Phi)$; see [15, Lemma 23] and [17, Lemma 6.1]. ∎

**3.4. A quasi-best approximation result for the Scott-Zhang operator.** In this section, we prove that the Scott–Zhang interpolation operator $\Pi_h$ satisfies the following local quasi-best approximation property. In doing so, we adapt the proof of [17, Theorem 5.7] to our weighted setting.

THEOREM 3.4 (local approximation). *Let $\Phi$ be a given $N$-function that satisfies the conditions in Assumption 2.1, let $\omega \in A_1$, and let $v \in W^{1,\Phi}(\omega,\Omega)$. Then, for each $T \in \mathscr{T}_h$ we have the bound*

$$\fint_T \omega(x)|\mathbf{V}(\nabla v(x)) - \mathbf{V}(\nabla \Pi_h v(x))|^2 \mathrm{d}x \lesssim \inf_{\mathbf{q}\in\mathbb{R}^d} \fint_{S_T} \omega(x)|\mathbf{V}(\nabla v(x)) - \mathbf{V}(\mathbf{q})|^2 \mathrm{d}x, \tag{3.8}$$

*where the hidden constant depends on $\sigma$, $\Delta_2(\Phi)$, and $[\omega]_{A_1}$. In particular, if $\nabla \mathbf{V}(\nabla v)$ belongs to $L^2(\omega,\Omega)$, then we have the bound*

$$\fint_T \omega(x)|\mathbf{V}(\nabla v(x)) - \mathbf{V}(\nabla \Pi_h v(x))|^2 \mathrm{d}x \lesssim h_T^2 \fint_{S_T} \omega(x)|\nabla \mathbf{V}(\nabla v)(x)|^2 \mathrm{d}x, \tag{3.9}$$

*where the hidden constant depends on $\sigma$, $\Delta_2(\Phi)$, and $[\omega]_{A_1}$.*

*Proof.* We begin by noting that the equivalence (2.15) allows us to obtain

$$\int_\Omega \omega(x)|\mathbf{V}(\nabla v(x))|^2 \mathrm{d}x \leq C \int_\Omega \omega(x)\Phi(|\nabla v(x)|)\mathrm{d}x < \infty$$

because $v \in W^{1,\Phi}(\omega,\Omega)$. This implies that $\mathbf{V}(\nabla v) \in L^2(\omega,\Omega)$.

We now let $\mathbf{q} \in \mathbb{R}^d$ be arbitrary and apply the triangle inequality to obtain

$$\begin{aligned}\fint_T \omega(x)|\mathbf{V}(\nabla v(x)) - \mathbf{V}(\nabla \Pi_h v(x))|^2 \mathrm{d}x &\lesssim \fint_T \omega(x)|\mathbf{V}(\nabla v(x)) - \mathbf{V}(\mathbf{q})|^2 \mathrm{d}x \\ &\quad + \fint_T \omega(x)|\mathbf{V}(\mathbf{q}) - \mathbf{V}(\nabla \Pi_h v(x))|^2 \mathrm{d}x =: \mathbf{I}_T + \mathbf{II}_T.\end{aligned}$$

Let $\mathfrak{p} \in \mathbb{P}_1(S_T)$ be such that $\nabla \mathfrak{p} = \mathbf{q}$. Note that $\mathbf{q} = \nabla \mathfrak{p} = \nabla \Pi_h \mathfrak{p}$ because $\Pi_h \mathfrak{p} = \mathfrak{p}$. We again use the equivalence (2.14) and the previous observation to bound the term $\mathbf{II}_T$ as follows:

$$\begin{aligned}\fint_T \omega(x)|\mathbf{V}(\mathbf{q}) - \mathbf{V}(\nabla \Pi_h v(x))|^2 \mathrm{d}x &\lesssim \fint_T \omega(x)\Phi_{|\mathbf{q}|}(|\mathbf{q} - \nabla \Pi_h v(x)|)\mathrm{d}x \\ &= C \fint_T \omega(x)\Phi_{|\mathbf{q}|}(|\nabla \Pi_h \mathfrak{p}(x) - \nabla \Pi_h v(x)|)\mathrm{d}x \\ &= C \fint_T \omega(x)\Phi_{|\mathbf{q}|}(|\nabla \Pi_h (\mathfrak{p} - v)(x)|)\mathrm{d}x.\end{aligned}$$

We now apply the stability estimate of Corollary 3.3 with $a = |\mathbf{q}|$ to the function $\mathfrak{p} - v$ and arrive at

$$\fint_T \omega(x)|\mathbf{V}(\mathbf{q}) - \mathbf{V}(\nabla \Pi_h v(x))|^2 \mathrm{d}x \lesssim \fint_{S_T} \omega(x)\Phi_{|\mathbf{q}|}(|\mathbf{q} - \nabla v(x)|)\mathrm{d}x.$$

As a final step, we apply the equivalence (2.14) once again to derive a bound for $\mathbf{II}_T$:

$$\mathbf{II}_T \lesssim \fint_{S_T} \omega(x)\Phi_{|\mathbf{q}|}(|\mathbf{q} - \nabla v(x)|)\mathrm{d}x \leq C \fint_{S_T} \omega(x)|\mathbf{V}(\mathbf{q}) - \mathbf{V}(\nabla v(x))|^2 \mathrm{d}x.$$

In what follows, we bound $\mathbf{I}_T$. To do this, we use that $T \subset S_T$ with $|T| \simeq |S_T|$ to obtain

$$\mathbf{I}_T \leq C \fint_{S_T} \omega(x)|\mathbf{V}(\mathbf{q}) - \mathbf{V}(\nabla v(x))|^2 \mathrm{d}x.$$

A combination of the bounds derived for $\mathbf{I}_T$ and $\mathbf{II}_T$ allows us to conclude for every $\mathbf{q} \in \mathbb{R}^d$ that

$$\fint_T \omega(x)|\mathbf{V}(\nabla v(x)) - \mathbf{V}(\nabla \Pi_h v(x))|^2 \mathrm{d}x \leq C \fint_{S_T} \omega(x)|\mathbf{V}(\mathbf{q}) - \mathbf{V}(\nabla v(x))|^2 \mathrm{d}x. \tag{3.10}$$

This estimate immediately yields (3.8).

After we have derived the local approximation result (3.10), the a priori error estimate (3.9) follows from the weighted Poincaré inequality (2.23). Note that (3.10) holds for every $\mathbf{q} \in \mathbb{R}^d$ and that $\mathbf{V} : \mathbb{R}^d \to \mathbb{R}^d$ is surjective. This concludes the proof. ∎

**3.5. An a priori error bound.** We conclude with the following a priori error estimate, which is the main result of this section.

THEOREM 3.5 (a priori error bound). *Let $u \in W_0^{1,\varphi}(\Omega)$ be the weak solution of (3.1), and let $u_h \in \mathbb{V}_h$ be its finite element approximation defined in (3.2), where $\varphi$ satisfies the conditions in Assumption 2.1. If $\nabla \mathbf{V}(\nabla u) \in L^2(\omega, \Omega)$, then*

$$\|\mathbf{V}(\nabla u) - \mathbf{V}(\nabla u_h)\|_{L^2(\omega,\Omega)} \lesssim h \|\nabla \mathbf{V}(\nabla u)\|_{L^2(\omega,\Omega)}, \tag{3.11}$$

*where the hidden constant depends on $\sigma$, $\Delta_2(\Phi)$, and $[\omega]_{A_1}$. Recall that $\omega \in A_1$ and that $\varphi$ is an $N$-function that satisfies the conditions in Assumption 2.1.*

*Proof.* The error bound (3.11) follows from the quasi-best approximation property (3.4) in conjunction with the the error estimate (3.9). ∎

*Remark* 3.6 (regularity). Note that it is fundamental to the error estimate of Theorem 3.5 that $\nabla \mathbf{V}(\nabla u) \in L^2(\omega, \Omega)$. In the unweighted setting, i.e., $\omega \equiv 1$, such results are proved in [1, 7, 8, 15, 18].

**4. The obstacle problem.** We now consider the constrained minimization of the functional $\mathcal{J}$ defined in (1.1). In the same setting and with the same assumptions as in Section 3, we additionally assume that we are given $\psi \in C^2(\bar{\Omega})$ with $\psi \leq 0$ on $\partial\Omega$. We define the admissible set

$$\mathcal{K}_\psi := \left\{ w \in W_0^{1,\varphi}(\omega, \Omega) : w \geq \psi \ a.e. \ in \ \Omega \right\}. \tag{4.1}$$

Recall that $\omega \in A_1$ and that $\varphi$ is an $N$-function that satisfies the conditions in Assumption 2.1. The problem that interests us is to find $u \in \mathcal{K}_\psi$ such that

$$\mathcal{J}(u) \leq \mathcal{J}(v) \qquad \forall v \in \mathcal{K}_\psi.$$

A standard convex minimization argument (see also [21, 27]) shows that the solution to this problem can be equivalently characterized as the following variational inequality. Find $u \in \mathcal{K}_\psi$ such that

$$\int_\Omega \omega(x)\mathbf{A}(\nabla u(x)) \cdot \nabla(u-v)(x)\,\mathrm{d}x \leq \int_\Omega \omega(x) f(x)(u-v)(x)\,\mathrm{d}x \quad \forall v \in \mathcal{K}_\psi. \tag{4.2}$$

Since it will be useful for what follows, we define $\lambda \in \left(W^{1,\varphi}(\omega,\Omega)\right)'$ as

$$\langle \lambda, w\rangle = \int_\Omega \omega(x)\mathbf{A}(\nabla u(x)) \cdot \nabla w(x)\,\mathrm{d}x - \int_\Omega \omega(x) f(x) w(x)\,\mathrm{d}x. \tag{4.3}$$

Note, then, that (4.2) can be rewritten as follows: Find $u \in \mathcal{K}_\psi$ such that

$$\langle \lambda, u - v\rangle \leq 0 \quad \forall v \in \mathcal{K}_\psi.$$

The following result describes further properties of $\lambda$.

PROPOSITION 4.1 (properties of $\lambda$). *The functional $\lambda$ defined in (4.3) defines a nonnegative Radon measure. In addition, $\lambda$ also satisfies the following properties:*

- *For every $\phi \in C_0^\infty(\Omega)$ that is nonnegative, i.e., $\phi \geq 0$ in $\Omega$, we have*

$$\langle \lambda, \phi\rangle \geq 0.$$

- *The following* complementarity condition *holds:*

$$\langle \lambda, u - \psi\rangle = 0.$$

*Proof.* Let $\phi \in C_0^\infty(\Omega)$ with $\phi \geq 0$ in $\Omega$. Define $v = u + \phi \in W_0^{1,\varphi}(\omega,\Omega)$. Since $v$ satisfies $v \geq u \geq \psi$ in $\Omega$, $v$ is admissible in (4.2). We can therefore substitute $v$ in the variational inequality (4.2) and obtain

$$0 \geq \langle \lambda, u - (u+\phi)\rangle = -\langle \lambda, \phi\rangle.$$

This proves the first property. We also note that this property shows that $\lambda$ defines a nonnegative distribution. The Riesz-Schwartz theorem (see [41, Chapter 1, §4, Théorème V] and [44, Chapter 1, Sec. 1.7, Theorem II]) then shows that $\lambda$ is a nonnegative Radon measure.

To obtain the second property, we define the coincidence and noncoincidence sets

$$\mathcal{C} = \{x \in \Omega : u(x) = \psi(x)\}, \qquad \Omega^+ = \Omega \setminus \mathcal{C} = \{x \in \Omega : u(x) > \psi(x)\}.$$

With these sets at hand, we can then write

$$\langle \lambda, u-\psi\rangle = \int_{\mathcal{C}} (u-\psi)(x)\,\mathrm{d}\lambda(x) + \int_{\Omega^+} (u-\psi)(x)\,\mathrm{d}\lambda(x) = \int_{\Omega^+} (u-\psi)(x)\,\mathrm{d}\lambda(x). \tag{4.4}$$

Now let $\mathcal{N} \subset \Omega^+$ be open, and let $\phi \in C_0^\infty(\mathcal{N})$. For $\epsilon > 0$, but sufficiently small, we have that the function $v = u - \epsilon\phi \in W_0^{1,\varphi}(\omega,\Omega)$ belongs to $\mathcal{K}_\psi$. In fact, we have that $u(x) > \psi(x)$ up to a null set on $\Omega^+$. Then, for a sufficiently small $\epsilon > 0$, we obtain that $u(x) - \epsilon\phi(x) \geq \psi(x)$ in $\Omega$. If we replace this function $v$ in (4.2), we obtain

$$0 \geq \langle \lambda, u - (u - \epsilon\phi)\rangle = \epsilon\langle \lambda, \phi\rangle.$$

From this and the property that $\langle \lambda, \phi\rangle \geq 0$ for every $\phi \in C_0^\infty(\Omega)$ that is nonnegative, we conclude that $\langle \lambda, \phi\rangle = 0$ for every $\phi \in C_0^\infty(\Omega^+)$ that is nonnegative, i.e.,

$$\int_{\Omega^+} \phi(x)\,\mathrm{d}\lambda(x) = 0, \qquad \forall \phi \in C_0^\infty(\Omega^+) : \ \phi \geq 0 \text{ in } \Omega^+.$$

Since $u - \psi > 0$ in $\Omega^+$, an approximation argument and (4.4) then show that $\langle \lambda, u - \psi\rangle = 0$ as claimed. □

**4.1. Finite element discretization.** To approximate the solution of (4.2), we retain the notation and constructions from Section 3.1. In addition, we need to introduce the positivity preserving interpolant $\wp_h : L^1(\Omega) \to \mathbb{V}_h$, which was originally developed in [12]. Let $\mathscr{V} = \{\mathtt{v}\}$ be the collection of vertices of $\mathscr{T}_h$ and let $\mathscr{V}_{\text{in}} \coloneqq \mathscr{V} \cap \Omega$. Recall that since $\mathbb{V}_h$ consists of piecewise linears, there is a bijection between $\mathscr{V}_{\text{in}}$ and the canonical Lagrange basis $\{\phi_{\mathtt{v}}\}_{\mathtt{v}\in\mathscr{V}_{\text{in}}}$ of $\mathbb{V}_h$. Given $\mathtt{v} \in \mathscr{V}_{\text{in}}$, we let

$$S_{\mathtt{v}} = \bigcup \{T \in \mathscr{T}_h : T \ni \mathtt{v}\} .$$

For $\mathtt{v} \in \mathscr{V}_{\text{in}}$, we define $\mathcal{B}_{\mathtt{v}}$ to be the largest ball centered in $\mathtt{v}$ and contained in $S_{\mathtt{v}}$. The positivity preserving interpolant $\wp_h$ is defined as follows: for $w \in L^1(\Omega)$,

$$\wp_h w = \sum_{\mathtt{v}\in\mathscr{V}_{\text{in}}} \left( \fint_{\mathcal{B}_{\mathtt{v}}} w(x)\,\mathrm{d}x \right) \phi_{\mathtt{v}}. \tag{4.5}$$

Clearly, $w \geq 0$ in $\Omega$ implies $\wp_h w \geq 0$ in $\Omega$. Further properties will be detailed below.

With the interpolant $\wp_h$ at hand, we can define the discrete admissible set

$$\mathcal{K}_{h,\psi} \coloneqq \{w_h \in \mathbb{V}_h : w_h \geq \wp_h \psi \text{ in } \Omega\} . \tag{4.6}$$

The finite element approximation of the solution to (4.2) is $u_h \in \mathcal{K}_{h,\psi}$, which satisfies the following variational inequality for every $v_h \in \mathcal{K}_{h,\psi}$:

$$\int_\Omega \omega(x)\mathbf{A}(\nabla u_h(x)) \cdot \nabla(u_h - v_h)(x)\,\mathrm{d}x \leq \int_\Omega \omega(x) f(x)(u_h - v_h)(x)\,\mathrm{d}x. \tag{4.7}$$

Standard arguments of variational inequalities yield the existence and uniqueness of a solution to (4.7).

**4.2. A priori error bounds.** We now analyze error bounds for our scheme.

PROPOSITION 4.2 (first error bound). *Let $u \in \mathcal{K}_\psi$ and $u_h \in \mathcal{K}_{h,\psi}$ be the solutions of (4.2) and (4.7), respectively. If $u \in L^2(\omega,\Omega)$ and $\lambda \in L^2(\omega^{-1},\Omega)$, then we have*

$$\begin{aligned}\|\mathbf{V}(\nabla u) - \mathbf{V}(\nabla u_h)\|^2_{L^2(\omega,\Omega)} &\lesssim \|\lambda\|_{L^2(\omega^{-1},\Omega)} \| [u-\psi] - \wp_h [u-\psi] \|_{L^2(\omega,\Omega)} \\ &\quad + \int_\Omega \omega(x) |\mathbf{V}(\nabla u(x)) - \mathbf{V}(\nabla \wp_h u(x))|^2\,\mathrm{d}x,\end{aligned}$$

*where $\wp_h$ is the positivity preserving interpolant defined in (4.5). In the previous estimate, the hidden constant is independent of $h$.*

*Proof.* We begin the proof by setting some notation. We define the error $e = u - u_h$ and the auxiliary function $z = u - \wp_h u$. Note that $\wp_h u \in \mathcal{K}_{h,\psi}$.

With the help of (2.13) we can now derive the following bound:

$$\|\mathbf{V}(\nabla u) - \mathbf{V}(\nabla u_h)\|^2_{L^2(\omega,\Omega)} \lesssim \int_\Omega \omega(x) \left[\mathbf{A}(\nabla u(x)) - \mathbf{A}(\nabla u_h(x))\right] \cdot \nabla e(x)\,\mathrm{d}x =: \mathfrak{D},$$

which we further decompose as

$$\begin{aligned}\mathfrak{D} &= \int_\Omega \omega(x) \left[\mathbf{A}(\nabla u(x)) - \mathbf{A}(\nabla u_h(x))\right] \cdot \nabla z(x)\,\mathrm{d}x \\ &\quad + \int_\Omega \omega(x) \left[\mathbf{A}(\nabla u(x)) - \mathbf{A}(\nabla u_h(x))\right] \cdot \nabla(\wp_h u - u_h)(x)\,\mathrm{d}x =: \mathfrak{L} + \mathfrak{N}.\end{aligned}$$

Let us first bound the term $\mathfrak{N}$. To do this, we use the discrete variational inequality (4.7) and obtain

$$\begin{aligned}\mathfrak{N} &= \int_\Omega \omega(x)\mathbf{A}(\nabla u(x))\cdot\nabla(\wp_h u - u_h)(x)\,\mathrm{d}x \\ &\quad + \int_\Omega \omega(x)\mathbf{A}(\nabla u_h(x))\cdot\nabla(u_h - \wp_h u)(x)\,\mathrm{d}x \\ &\leq \int_\Omega \omega(x)\mathbf{A}(\nabla u(x))\cdot\nabla(\wp_h u - u_h)(x)\,\mathrm{d}x + \int_\Omega \omega(x) f(x)(u_h - \wp_h u)(x)\,\mathrm{d}x \\ &= \langle \lambda, \wp_h u - u_h\rangle,\end{aligned}$$

where $\lambda$ is defined in (4.3). We can then continue and write

$$\begin{aligned}\mathfrak{N} &\leq \langle \lambda, u-\psi\rangle + \langle\lambda, \wp_h\psi - u_h\rangle + \langle \lambda, [\wp_h u - \wp_h \psi] - [u-\psi]\rangle \\ &= \langle\lambda, \wp_h\psi - u_h\rangle + \langle \lambda, [\wp_h u - \wp_h \psi] - [u-\psi]\rangle\,,\end{aligned}$$

where we have used the complementarity condition $\langle \lambda, u-\psi\rangle = 0$ from Proposition 4.1. Let us now examine each of the remaining terms separately.

- Since $u_h \in \mathcal{K}_{h,\psi}$, we have $u_h \geq \wp_h\psi$. This and the fact that $\lambda$ is a nonnegative Radon measure (see Proposition 4.1) allows us to obtain

$$\langle\lambda, \wp_h\psi - u_h\rangle = \int_\Omega (\wp_h\psi - u_h)(x)\,\mathrm{d}\lambda(x) \leq 0.$$

- Since we have assumed that $\lambda \in L^2(\omega^{-1},\Omega)$, the fact that $\wp_h$ is linear shows that

$$\begin{aligned}\langle \lambda, [\wp_h u - \wp_h\psi] - [u-\psi]\rangle &= \langle \lambda, \wp_h\,[u-\psi] - [u-\psi]\rangle \\ &\leq \|\lambda\|_{L^2(\omega^{-1},\Omega)} \|\,[u-\psi] - \wp_h\,[u-\psi]\,\|_{L^2(\omega,\Omega)},\end{aligned}$$

  To obtain the last estimate, we used Hölder's inequality.

To summarize, we have come to the following conclusion:

$$\mathfrak{N} \leq \|\lambda\|_{L^2(\omega^{-1},\Omega)} \|\,[u-\psi] - \wp_h\,[u-\psi]\,\|_{L^2(\omega,\Omega)}.$$

To estimate the term $\mathfrak{L}$, we can repeat the arguments we used to control the term $\mathfrak{I}$ in the proof of Theorem 3.1. These arguments lead us to the conclusion that

$$\mathfrak{L} \leq C\delta \|\mathbf{V}(\nabla u) - \mathbf{V}(\nabla u_h)\|^2_{L^2(\omega,\Omega)} + C_\delta \int_\Omega \omega(x)\varphi_{|\nabla u(x)|}(|\nabla z(x)|)\,\mathrm{d}x.$$

If we substitute this estimate and the one obtained for $\mathfrak{N}$ in the bound in which $\mathfrak{D}$ is defined, we obtain

$$\begin{aligned}(4.8)\quad \|\mathbf{V}(\nabla u) - \mathbf{V}(\nabla u_h)\|^2_{L^2(\omega,\Omega)} &\leq C\delta\|\mathbf{V}(\nabla u) - \mathbf{V}(\nabla u_h)\|^2_{L^2(\omega,\Omega)} \\ + C_\delta \int_\Omega \omega(x)\varphi_{|\nabla u(x)|}(|\nabla z(x)|)\,\mathrm{d}x &+ C\|\lambda\|_{L^2(\omega^{-1},\Omega)}\|\,[u-\psi] - \wp_h\,[u-\psi]\,\|_{L^2(\omega,\Omega)}.\end{aligned}$$

The term $\|\mathbf{V}(\nabla u) - \mathbf{V}(\nabla u_h)\|^2_{L^2(\omega,\Omega)}$ appearing in the previous inequality can be absorbed on the left hand side if $\delta$ is chosen carefully. This shows that

$$\begin{aligned}(4.9)\quad \|\mathbf{V}(\nabla u) - \mathbf{V}(\nabla u_h)\|^2_{L^2(\omega,\Omega)} &\lesssim \|\lambda\|_{L^2(\omega^{-1},\Omega)}\|\,[u-\chi] - \wp_h\,[u-\chi]\,\|_{L^2(\omega,\Omega)} \\ &\quad + \int_\Omega \omega(x)\varphi_{|\nabla u(x)|}(|\nabla z(x)|)\,\mathrm{d}x,\end{aligned}$$

with $z = u - \wp_h u$. Use the equivalences (2.13)–(2.14) one last time to obtain the assertion. This concludes the proof. ∎

**4.3. The positivity preserving interpolant.** Note that up to this point, no property other than positivity has been required of $\wp_h$. In what follows, however, the stability and approximation properties of $\wp_h$ are required. Ideally, we would like to have an operator that is a projection and satisfies (3.5). Note, however, that $\wp_h$ is not a projection. In fact, it is not possible to construct a positivity preserving projection; see [38]. For this reason, we need to prove suitable properties for $\wp_h$.

LEMMA 4.3 (properties of $\wp_h$). *The interpolant $\wp_h$ defined in (4.5) satisfies the following properties.*

1. *$\wp_h$ is positivity preserving: If $w \geq 0$ in $\Omega$, then $\wp_h w \geq 0$ in $\Omega$.*
2. *$\wp_h$ is symmetric in the sense that: If $\mathtt{v} \in \mathscr{V}_{\text{in}}$ and $w_{|\mathcal{B}_\mathtt{v}} \in \mathbb{P}_1$, then $\wp_h w(\mathtt{v}) = w(\mathtt{v})$.*
3. *$\wp_h$ is locally invariant: For every $T \in \mathscr{T}_h$ with $T \cap \partial\Omega = \emptyset$, if $w_{|S_T} \in \mathbb{P}_1$, then $\wp_h w_{|T} = w_{|T}$.*
4. *$\wp_h$ is locally stable: For all $p \in [1,\infty]$ and every $T \in \mathscr{T}_h$, we have*

$$\|\wp_h w\|_{L^p(T)} \lesssim \|w\|_{L^p(S_T)} \quad \forall w \in L^p(\Omega),$$
$$\|\nabla \wp_h w\|_{L^p(T)} \lesssim \|\nabla w\|_{L^p(S_T)} \quad \forall w \in W_0^{1,p}(\Omega).$$

5. *$\wp_h$ is weighted Orlicz stable: Let $\omega \in A_1$ and let $\Phi$ be an $N$-function that satisfies the conditions in Assumption 2.1. Then, for every $T \in \mathscr{T}_h$ and for all $a \geq 0$,*

$$\fint_T \omega(x)\Phi_a(|\nabla \wp_h w(x)|)\,\mathrm{d}x \lesssim \fint_{S_T} \omega(x)\Phi_a(|\nabla w(x)|)\,\mathrm{d}x,$$

*where the hidden constant does not depend on $T$ or $a$.*

6. *$\wp_h$ has weighted approximation properties: If $w \in W_0^{1,2}(\omega,\Omega)$, then, for all $T \in \mathscr{T}_h$,*

$$\|w - \wp_h w\|_{L^2(\omega,T)} \lesssim h_T \|\nabla w\|_{L^2(\omega,S_T)}.$$

*Proof.* We examine each statement separately.

1. This property follows directly from the definition of $\wp_h$.
2. Since $w_{|\mathcal{B}_\mathtt{v}} \in \mathbb{P}_1$, we can write $w(x) = a + \mathbf{b}\cdot(x - \mathtt{v})$ for some $a \in \mathbb{R}$ and $\mathbf{b} \in \mathbb{R}^d$. Then, we have

$$\wp_h w(\mathtt{v}) = \fint_{\mathcal{B}_\mathtt{v}} [a + \mathbf{b}\cdot(x-\mathtt{v})]\,\mathrm{d}x = a + \mathbf{b}\cdot\fint_{\mathcal{B}_\mathtt{v}} (x-\mathtt{v})\,\mathrm{d}x = a = w(\mathtt{v}).$$

3. This follows from repeating the previous computation on every vertex of $T$, which forms an unisolvent set for $\mathbb{P}_1(T)$.
4. This is proved in [12, Lemma 3.1].
5. This follows from the preceding stability properties by repeating the arguments in the proofs of Theorem 3.2 and Corollary 3.3.
6. Let $T \in \mathscr{T}_h$, and let $w \in W_0^{1,2}(\omega,\Omega)$. Let $\mathfrak{p} \in \mathbb{P}_1(\mathbb{R}^d)$ be arbitrary. We begin the proof by noting that if we can show the bound

$$\|\wp_h w - \mathfrak{p}\|_{L^2(\omega,T)} \lesssim \|w - \mathfrak{p}\|_{L^2(\omega,S_T)} + h_T\|\nabla(w-\mathfrak{p})\|_{L^2(\omega,S_T)}, \tag{4.10}$$

then a simple application of the triangle inequality shows

$$\|w - \wp_h w\|_{L^2(\omega,T)} \lesssim \|w - \mathfrak{p}\|_{L^2(\omega,S_T)} + h_T\|\nabla(w-\mathfrak{p})\|_{L^2(\omega,S_T)}.$$

A suitable choice of $\mathfrak{p}$ then shall imply the thesis.

To prove (4.10), we proceed differently depending on whether $T$ is an interior element or $T$ touches the boundary. If $T$ is an interior element, i.e., $T \cap \partial\Omega = \emptyset$, then local invariance and stability imply the result. On the other hand, if the element $T$ touches the boundary we proceed as follows. Let $\{\mathtt{v}_i^T\}_{i=0}^d$ be the collection of vertices of $T$. Since $(\wp_h w - \mathfrak{p})_{|T} \in \mathbb{P}_1$, we can write

$$(\wp_h w - \mathfrak{p})_{|T} = \sum_{i=0}^{d} (\wp_h w - \mathfrak{p})(\mathtt{v}_i^T)\phi_{\mathtt{v}_i^T}.$$

We now use that $\|\phi_{\mathtt{v}}\|_{L^\infty(\Omega)} = 1$ for all $\mathtt{v} \in \mathscr{V}_{\rm in}$ to obtain the estimate

$$\|\wp_h w - \mathfrak{p}\|_{L^2(\omega,T)}^2 \lesssim \max_{i=0}^{d} \left|(\wp_h w - \mathfrak{p})(\mathtt{v}_i^T)\right|^2 \omega(T).$$

Now we distinguish two cases. If $\mathtt{v}_i^T \in \mathscr{V}_{\rm in}$, then

$$\begin{aligned}\left|(\wp_h w - \mathfrak{p})(\mathtt{v}_i^T)\right| = \left|\wp_h (w - \mathfrak{p})(\mathtt{v}_i^T)\right| &\leq \fint_{\mathcal{B}_{\mathtt{v}_i^T}} |w(x) - \mathfrak{p}(x)| \,\mathrm{d}x \\ &\lesssim h_T^{-d} \omega^{-1}(T)^{\frac12} \|w - \mathfrak{p}\|_{L^2(\omega,S_T)}.\end{aligned}$$

On the other hand, if $\mathtt{v}_i^T \in \partial\Omega$, then $\wp_h w(\mathtt{v}_i^T) = 0$ by definition. Let $F \subset T$ be the $(d-1)$-dimensional subsimplex containing $\mathtt{v}_i^T$. We thus have that

$$\left|(\wp_h w - \mathfrak{p})(\mathtt{v}_i^T)\right| = |\mathfrak{p}(\mathtt{v}_i^T)| \lesssim \fint_F |\mathfrak{p}(x)| \,\mathrm{d}x = \fint_F |w(x) - \mathfrak{p}(x)| \,\mathrm{d}x,$$

where we have used that $w_{|F} = 0$. The scaled trace inequality (2.24) then yields

$$\begin{aligned}\fint_F |w(x) - \mathfrak{p}(x)| \,\mathrm{d}x &\lesssim \fint_{S_T} |w(x) - \mathfrak{p}(x)| \,\mathrm{d}x + h_T \fint_{S_T} |\nabla (w - \mathfrak{p})(x)| \,\mathrm{d}x \\ &\leq h_T^{-d} \omega^{-1}(S_T)^{\frac12} \left[\|w - \mathfrak{p}\|_{L^2(\omega,S_T)} + h_T \|\nabla (w - \mathfrak{p})\|_{L^2(\omega,S_T)}\right].\end{aligned}$$

To summarize, we have arrived at the following estimate:

$$\begin{aligned}\|\wp_h w - \mathfrak{p}\|_{L^2(\omega,T)}^2 &\lesssim \frac{\omega(T)\omega^{-1}(S_T)}{h_T^{2d}} \left[\|w - \mathfrak{p}\|_{L^2(\omega,S_T)} + h_T \|\nabla (w - \mathfrak{p})\|_{L^2(\omega,S_T)}\right]^2 \\ &\lesssim [\omega]_{A_2} \left[\|w - \mathfrak{p}\|_{L^2(\omega,S_T)} + h_T \|\nabla (w - \mathfrak{p})\|_{L^2(\omega,S_T)}\right]^2.\end{aligned}$$

All properties have been proved. This concludes the proof. ∎

Next, we obtain an analogue of the bound (3.8).

PROPOSITION 4.4 (interpolation error). *Let $w \in W_0^{1,\varphi}(\omega,\Omega)$ verify $\nabla \mathbf{V}(\nabla w) \in L^2(\omega,\Omega)$. Assume that $\mathscr{T}_h = \{T\}$ is such that no simplex $T$ has more than one $(d-1)$-dimensional subsimplex on $\partial\Omega$ and that $\omega \in A_1(\Omega)$. If $h$ is sufficiently small, then*

$$\|\mathbf{V}(\nabla w) - \mathbf{V}(\nabla \wp_h w)\|_{L^2(\omega,\Omega)} \lesssim h \|\nabla \mathbf{V}(\nabla w)\|_{L^2(\omega,\Omega)},$$

*where the hidden constant depends on $[\omega]_{A_1}$ and $\|\omega\|_{L^\infty(\mathcal{G})}$ but it is independent of $h$. Recall that $\varphi$ is an $N$-function that satisfies the conditions in Assumption 2.1.*

*Proof.* We begin by noting that for every $T \in \mathscr{T}_h$ and every $\mathfrak{p} \in \mathbb{P}_1(\mathbb{R}^d)$, we have

$$\int_T \omega(x)|\mathbf{V}(\nabla w(x)) - \mathbf{V}(\nabla \wp_h w)|^2 \,\mathrm{d}x \lesssim \int_T \omega(x)|\mathbf{V}(\nabla w(x)) - \mathbf{V}(\nabla \mathfrak{p})|^2 \,\mathrm{d}x + \int_T \omega(x)|\mathbf{V}(\nabla \mathfrak{p}) - \mathbf{V}(\nabla \wp_h w)|^2 \,\mathrm{d}x =: \mathbf{I} + \mathbf{II}.$$

Define the sets $\mathscr{T}_h^{\mathrm{in}} := \{T \in \mathscr{T}_h : T \cap \partial\Omega = \emptyset\}$ and $\mathscr{T}_h^{\partial} := \{T \in \mathscr{T}_h : T \cap \partial\Omega \neq \emptyset\}$. We now partition $\mathscr{T}_h = \mathscr{T}_h^{\mathrm{in}} \sqcup \mathscr{T}_h^{\partial}$.

Let $T \in \mathscr{T}_h^{\mathrm{in}}$. The local invariance of $\wp_h$, proved in Lemma 4.3, implies $\wp_h \mathfrak{p} = \mathfrak{p}$. Using the equivalences (2.13)–(2.14) and the weighted Orlicz stability of $\wp_h$ derived in Lemma 4.3, we obtain the estimate

$$\mathbf{II} \lesssim \int_T \omega(x)\varphi_{|\nabla \mathfrak{p}|}(|\nabla \wp_h(w-\mathfrak{p})|) \,\mathrm{d}x \lesssim \int_{S_T} \omega(x)\varphi_{|\nabla \mathfrak{p}|}(|\nabla (w-\mathfrak{p})(x)| \,\mathrm{d}x \lesssim \int_{S_T} \omega(x)|\mathbf{V}(\nabla w(x)) - \mathbf{V}(\nabla \mathfrak{p})|^2 \,\mathrm{d}x.$$

As a result, for each $T \in \mathscr{T}_h^{\mathrm{in}}$ we have the bound

$$\int_T \omega(x)|\mathbf{V}(\nabla w(x)) - \mathbf{V}(\nabla \wp_h w)|^2 \,\mathrm{d}x \lesssim \int_{S_T} \omega(x)|\mathbf{V}(\nabla w(x)) - \mathbf{V}(\nabla \mathfrak{p})|^2 \,\mathrm{d}x,$$

where $\mathfrak{p} \in \mathbb{P}_1(\mathbb{R}^d)$ is arbitrary. Let $\mathbf{Q} \in \mathbb{R}^d$ and choose $\mathbf{z} = \nabla \mathfrak{p} \in \mathbb{R}^d$ so that $\mathbf{V}(\mathbf{z}) = \mathbf{Q}$. This is possible because $\mathbf{V}$ is surjective. This gives us, for every $T \in \mathscr{T}_h^{\mathrm{in}}$,

$$\int_T \omega(x)|\mathbf{V}(\nabla w(x)) - \mathbf{V}(\nabla \wp_h w)|^2 \,\mathrm{d}x \lesssim \int_{S_T} \omega(x)|\mathbf{V}(\nabla w(x)) - \mathbf{Q}|^2 \,\mathrm{d}x \lesssim h_T^2 \int_{S_T} \omega(x)|\nabla \mathbf{V}(\nabla w(x))|^2 \,\mathrm{d}x, \tag{4.11}$$

where we used the weighted Poincaré inequality (2.23) in the last step, by choosing the vector $\mathbf{Q}$ accordingly, and that by shape regularity $\mathrm{diam}(T) \simeq \mathrm{diam}(S_T)$.

We now consider $T \in \mathscr{T}_h^{\partial}$. If $h$ is sufficiently small, there is no fault in assuming that $T \subset \mathcal{G}$. Recall that $\mathcal{G}$ is as in Definition 2.4. Let $F \subset T$ be the unique $(d-1)$-dimensional subsimplex such that $F = T \cap \partial\Omega$ and denote by $\boldsymbol{\zeta}$ the unit outer normal to $T$ on $F$. Now let $\{\mathtt{v}_i^T\}_{i=0}^d$ be the set of vertices of $T$ and assume that $\{\mathtt{v}_i^T\}_{i=0}^d$ is numbered such that $F = \mathrm{conv}\{\mathtt{v}_i^T\}_{i=0}^{d-1}$. Define $\mathfrak{p} \in \mathbb{P}_1(\mathbb{R}^d)$ as $\mathfrak{p}(x) = b\boldsymbol{\zeta} \cdot (x - \mathtt{v}_0^T)$, where $b \in \mathbb{R}$ is to be chosen. We note that, by construction, $\mathfrak{p}_{|F} = 0$. Moreover, we have by definition that $\wp_h \mathfrak{p}_{|F} = 0$. Finally, the symmetry of $\wp_h$ proved in Lemma 4.3 implies that $\wp_h \mathfrak{p}(\mathtt{v}_d^T) = \mathfrak{p}(\mathtt{v}_d^T)$. From this follows that $\wp_h \mathfrak{p}_{|T} = \mathfrak{p}$.

Based on the constructions described in the previous paragraph, the weighted Orlicz stability of $\wp_h$ (see Lemma 4.3) allows us to treat $\mathbf{II}$ as before and obtain that, for every $T \in \mathscr{T}_h^{\partial}$,

$$\int_T \omega(x)|\mathbf{V}(\nabla w(x)) - \mathbf{V}(\nabla \wp_h w)|^2 \,\mathrm{d}x \lesssim \int_{S_T} \omega(x)|\mathbf{V}(\nabla w(x)) - \mathbf{V}(\nabla \mathfrak{p})|^2 \,\mathrm{d}x,$$

where $\mathfrak{p} \in \mathbb{P}_1(\mathbb{R}^d)$ is such that $\mathfrak{p}_{|F} = 0$, but is otherwise arbitrary.

Let us now use the fact that $w_{|F} = 0$ to deduce that $\nabla w$ and $\boldsymbol{\zeta}$ are parallel on $F$. This implies that $\mathbf{V}(\nabla w(x))_{|F} = v(x)\boldsymbol{\zeta}$ for some $v : F \to \mathbb{R}$. Now, given $x \in T$, we denote by $\hat{x} \in F$ the projection of $x$ onto $F$. We note that, due to shape regularity, $|\mathtt{v}_d^T - \hat{\mathtt{v}}_d^T| \simeq h_T$. Define $Q$ to be the prism with base $F$ and height $H := |\mathtt{v}_d^T - \hat{\mathtt{v}}_d^T|$. We may now extend $v$ to $Q$ via

$$v(x) = v(\hat{x}) \quad \forall x \in Q.$$

As a final preparatory step, we note that $\mathbf{V}(\nabla \mathfrak{p}) = \kappa \boldsymbol{\zeta}$, where $\kappa \in \mathbb{R}$ can be chosen arbitrarily by suitably specifying the value $\mathfrak{p}(\mathtt{v}_d^T)$. We now estimate

$$\mathbf{I} \lesssim \int_T \omega(x) |\mathbf{V}(\nabla w(x)) - v(x)\boldsymbol{\zeta}|^2 \,\mathrm{d}x + \int_T \omega(x)|v(x) - \kappa|^2 \,\mathrm{d}x =: \mathbf{I}_1 + \mathbf{I}_2.$$

In order to bound $\mathbf{I}_1$, we set some notation. We let $\mathbf{w}(x) = \mathbf{V}(\nabla w(x))$. Then,

$$\begin{aligned}\mathbf{I}_1 &\leq \int_Q \omega(x)|\mathbf{w}(x) - \mathbf{w}(\hat{x})|^2 \,\mathrm{d}x = \int_Q \omega(x) \left| \int_0^1 \nabla \mathbf{w}(tx + (1-t)\hat{x}) \cdot (x - \hat{x}) \right|^2 \,\mathrm{d}t \,\mathrm{d}x \\ &\lesssim \|\omega\|_{L^\infty(\mathcal{G})} \int_Q \int_0^1 |\nabla \mathbf{w}(tx + (1-t)\hat{x}) \cdot (x - \hat{x})|^2 \,\mathrm{d}t \,\mathrm{d}x,\end{aligned}$$

where we have used the fact that $\omega$ belongs to the reduced class $A_1(\Omega)$ (see Definition 2.4). We now introduce a coordinate system in which the first axis is aligned with $\boldsymbol{\zeta}$ and $Q$ is on the half space defined by $\{x \in \mathbb{R}^d : x_1 \geq 0\}$. We thus have that, if $x \in Q$, then $x = (x_1, x')^\top$ with $x_1 \geq 0$ and $x' \in F \subset \mathbb{R}^{d-1}$. We also note that $\hat{x} = (0, x')^\top$, $x - \hat{x} = (x_1, 0)^\top$, and $y = tx + (1-t)\hat{x} = (tx_1, x')^\top$ (see Figure 4.1). With these observations in mind, we continue the bound for the term $\mathbf{I}_1$ as follows:

$$\begin{aligned}\mathbf{I}_1 &\lesssim \|\omega\|_{L^\infty(\mathcal{G})} \int_Q \int_0^1 |\partial_1 \mathbf{w}(tx_1, x')|^2 |x_1|^2 \,\mathrm{d}t \,\mathrm{d}x \\ &= \|\omega\|_{L^\infty(\mathcal{G})} \int_0^H x_1^2 \int_F \int_0^1 |\partial_1 \mathbf{w}(tx_1, x')|^2 \,\mathrm{d}t \,\mathrm{d}x' \,\mathrm{d}x_1 \\ &= \|\omega\|_{L^\infty(\mathcal{G})} \int_0^H x_1 \int_F \int_0^{x_1} |\partial_1 \mathbf{w}(r, x')|^2 \,\mathrm{d}r \,\mathrm{d}x' \,\mathrm{d}x_1 \\ &\leq \|\omega\|_{L^\infty(\mathcal{G})} \int_0^H x_1 \int_F \int_0^H |\partial_1 \mathbf{w}(r, x')|^2 \,\mathrm{d}r \,\mathrm{d}x' \,\mathrm{d}x_1,\end{aligned}$$

where we have introduced the change of variables $r = tx_1$ in the innermost integral and used the fact that if $x \in Q$, then $x_1 \in [0, H]$. Now we just integrate with respect to $x_1$ and use that $H \simeq h_T$ to obtain

$$\begin{aligned}\mathbf{I}_1 &\lesssim \frac{1}{2} h_T^2 \|\omega\|_{L^\infty(\mathcal{G})} \int_F \int_0^H |\partial_1 \mathbf{w}(r, x')|^2 \,\mathrm{d}r \,\mathrm{d}x' \\ &\leq \frac{1}{2} h_T^2 \|\omega\|_{L^\infty(\mathcal{G})} \int_F \int_0^H |\nabla \mathbf{w}(r, x')|^2 \,\mathrm{d}r \,\mathrm{d}x' \\ &= \frac{1}{2} h_T^2 \|\omega\|_{L^\infty(\mathcal{G})} \int_Q |\nabla \mathbf{w}(y)|^2 \,\mathrm{d}y \\ &\leq \frac{1}{2} h_T^2 \|\omega\|_{L^\infty(\mathcal{G})} \|\omega^{-1}\|_{L^\infty(Q)} \int_Q \omega(y) |\nabla \mathbf{V}(\nabla w(y))|^2 \,\mathrm{d}y.\end{aligned}$$

We recall that $Q = [0, H] \times F$ and $\mathbf{w}(y) = \mathbf{V}(\nabla \mathbf{w}(y))$.

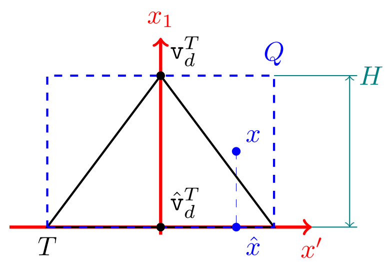


FIG. 4.1. *The geometric setting for the bound for $\mathbf{I}_1$ in the proof of Proposition 4.4. The element $T \in \mathscr{T}_h^\partial$ is shown in black, while the related prism is in dashed blue. We also show a generic point $x \in Q$ and its projection onto $F$, i.e., $\hat{x} \in F$.*

To bound $\mathbf{I}_2$, we recall that $H = |\mathrm{v}_d^T - \hat{\mathrm{v}}_d^T|$ and obtain that

$$
\begin{aligned}
\mathbf{I}_2 &\leq \|\omega\|_{L^\infty(\mathcal{G})} \int_0^H \int_F |v(\hat{x}) - \kappa|^2 \,\mathrm{d}\hat{x}\,\mathrm{d}t \lesssim h_T \|\omega\|_{L^\infty(\mathcal{G})} \int_F |\operatorname{tr}_F \mathbf{w}(\hat{x}) - \kappa\boldsymbol{\zeta}|^2 \,\mathrm{d}\hat{x} \\
&\lesssim h_T \|\omega\|_{L^\infty(\mathcal{G})} \left[ h_T^{-1} \int_T |\mathbf{w}(x) - \kappa\boldsymbol{\zeta}|^2 \,\mathrm{d}x + h_T \int_T |\nabla(\mathbf{w}(x) - \kappa\boldsymbol{\zeta})|^2 \,\mathrm{d}x \right] \\
&\lesssim \|\omega\|_{L^\infty(\mathcal{G})} \|\omega^{-1}\|_{L^\infty(T)} \left[ \int_T \omega(x) |\mathbf{w}(x) - \kappa\boldsymbol{\zeta}|^2 \,\mathrm{d}x + h_T^2 \int_T \omega(x) |\nabla \mathbf{w}(x)|^2 \,\mathrm{d}x \right],
\end{aligned}
$$

where we have used the scaled trace inequality (2.24). The weighted Poincaré inequality then shows that for every $T \in \mathscr{T}_h^\partial$,

$$
\int_T \omega(x) |\mathbf{V}(\nabla w(x)) - \mathbf{V}(\nabla \wp_h w)|^2 \,\mathrm{d}x \lesssim h_T^2 \int_{S_T} \omega(x) |\nabla \mathbf{V}(\nabla w(x))|^2 \,\mathrm{d}x, \tag{4.12}
$$

where the hidden constant depends on $[\omega]_{A_1}$ and $\|\omega\|_{L^\infty(\mathcal{G})}$.

It remains to add (4.11) and (4.12) and use shape regularity to obtain the claim. ∎

**4.4. Error estimate.** We are now ready to present an error estimate for the obstacle problem.

THEOREM 4.5 (error estimate). *Let $u \in W_0^{1,\varphi}(\omega, \Omega)$ and $u_h \in \mathbb{V}_h$ solve (4.2) and (4.7), respectively. Assume that $u, \mathbf{V}(\nabla u) \in W^{1,2}(\omega, \Omega)$, that $\lambda$, defined in (4.3), belongs to $L^2(\omega^{-1}, \Omega)$, and that $\omega \in A_1(\Omega)$. If $\mathscr{T}_h$ is such that no simplex has more than one $(d-1)$-dimensional subsimplex on $\partial\Omega$ and $h$ is sufficiently small, then*

$$
\|\mathbf{V}(\nabla u) - \mathbf{V}(\nabla u_h)\|_{L^2(\omega,\Omega)} \lesssim h \|\nabla \mathbf{V}(\nabla u)\|_{L^2(\omega,\Omega)} + h^{\frac{1}{2}} \|\nabla u\|_{L^2(\omega,\Omega)},
$$

*where the hidden constant is independent of $h$.*

*Proof.* We apply Proposition 4.4 to the conclusion of Proposition 4.2 and obtain

$$
\|\mathbf{V}(\nabla u) - \mathbf{V}(\nabla u_h)\|_{L^2(\omega,\Omega)}^2 \lesssim h^2 \|\nabla \mathbf{V}(\nabla u)\|_{L^2(\omega,\Omega)}^2 + \|[u - \psi] - \wp_h [u - \psi]\|_{L^2(\omega,\Omega)}.
$$

It remains to invoke the approximation property of the interpolant $\wp_h$, which was proved in Lemma 4.3, in order to be able to conclude. ∎

*Remark* 4.6 (regularity). Note that the error estimate depends on the assumption that $\mathbf{V}(\nabla u) \in W^{1,2}(\omega, \Omega)$. In addition, we need the regularity assumption $\lambda \in$

$L^2(\omega^{-1}, \Omega)$. We leave the exploration of these properties as open conjectures and refer the reader to [9, 10, 20, 25, 29, 45] for some results in this direction. We also note that the assumption $\nabla u \in L^2(\omega, \Omega)$ is not extremely restrictive. In the case of the $p$-Laplacian, for example, this reduces to the fact that $p \geq 2$. Indeed,

$$\int_\Omega |\nabla u(x)|^2 \omega(x) \, \mathrm{d}x \leq \left( \int_\Omega |\nabla u(x)|^p \omega(x) \, \mathrm{d}x \right)^{2/p} \left( \int_\Omega \omega(x) \, \mathrm{d}x \right)^{\frac{p-2}{p}}.$$

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